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The parametrix method approach to diffusions in a turbulent Gaussian environment

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Abstract

In this paper we deal with the solutions of Itô stochastic differential equation

$$\mathrm{d}X_\varepsilon(t) = \frac{1}{\varepsilon} \boldsymbol{V}\left(\frac{t}{\varepsilon^2}, \frac{X_\varepsilon(t)}{\varepsilon^\alpha}\right)\mathrm{d}t + \sqrt{2}\,\mathrm{d}\boldsymbol{B}(t),$$

for a small parameter ε. We prove that for $0 \leqslant \alpha < 1$ and $\boldsymbol{V}$ a divergence-free, Gaussian random field, sufficiently strongly mixing in t variable the family of processes $\{X_\varepsilon(t)\}_{t\geqslant 0}$, $\varepsilon > 0$ converges weakly to a Brownian motion. The entries of the covariance matrix of the limiting Brownian motion are given by $a_{i,j} = 2\delta_{i,j} + \int_{-\infty}^{+\infty} R_{i,j}(t,0)\,\mathrm{d}t$, $i, j = 1,\ldots,d$, where $[R_{i,j}(t,x)]$ is the covariance matrix of the field $\boldsymbol{V}$. To obtain this result we apply a version of the parametrix method for a linear parabolic PDE (see Friedman, 1963). © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Suppose that a particle is undergoing a diffusion with a drift induced by some external velocity field $\boldsymbol{V}(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. The equation describing such a motion, starting at the origin, can be written as

$$\mathrm{d}X(t) = \boldsymbol{V}(t, X(t))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}\boldsymbol{B}(t),$$

$$X(0) = 0.$$

Here $\boldsymbol{B}(t)$, $t \geqslant 0$ is a standard Brownian motion. In the case when the particle moves in a turbulent fluid we may assume that $\boldsymbol{V}$ is *a divergenceless*, *stationary*, *Gaussian random field* with strong mixing properties at, macroscopically, short temporal and

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spatial distances (see e.g. Chorin, 1994). Transforming coordinates, if necessary, we can also assume that the field is of *zero mean*.

Assume now that the velocity is of the form $\boldsymbol{V}(t,\varepsilon^\beta x)$, where ε is some small scaling parameter and $\beta \geqslant 0$. Introducing "macroscopic coordinates" t', x' by setting $t=t'/\varepsilon^2$ and $x=x'/\varepsilon$ we obtain $1/\varepsilon \boldsymbol{V}(t/\varepsilon^2, x/\varepsilon^{1-\beta})$ as the description of the field in the new coordinates (both here and in the sequel we omit writing primes for the new set of coordinates). The law of the trajectory in the macroscopic coordinates coincides with that of the solution of the equation

$$X_\varepsilon(t)=\frac{1}{\varepsilon}\int_0^t \boldsymbol{V}\left(\frac{s}{\varepsilon^2},\frac{X_\varepsilon(s)}{\varepsilon^\alpha}\right)\mathrm{d}s+\sqrt{2}\,\mathrm{d}\boldsymbol{B}(t), \tag{1}$$

where $\alpha = 1-\beta$. We wish to investigate the limiting behavior of processes $\{X_\varepsilon(t)\}_{t\geqslant 0}$, as $\varepsilon \downarrow 0$.

The case of *fields oscillating slowly in space*, i.e. when $\alpha = 0$, has been considered e.g. in Kunita (1990) (see Theorem 5.6.1 p. 264). It has been proven that if $\boldsymbol{V}$ is sufficiently strongly mixing then $\{X_\varepsilon(t)\}_{t\geqslant 0}$ converge weakly as $\varepsilon\downarrow 0$ to a Brownian motion whose covariance matrix equals $\boldsymbol{A}=[a_{i,j}]$, where

$$a_{i,j}=2\delta_{i,j}+\int_{-\infty}^{+\infty} R_{i,j}(t,0)\,\mathrm{d}t \tag{2}$$

and $[R_{i,j}]$ is the covariance matrix of $\boldsymbol{V}$. Similar results were also proven for fields of the form $(1/\varepsilon)\boldsymbol{V}(Z(t/\varepsilon^2),x)$, where $Z(t/\varepsilon^2)$ is a finite-dimensional ergodic noise (cf. Bouc and Pardoux, 1984; Kushner and Huang, 1985). The case of Ornstein–Uhlenbeck fields (i.e. Gaussian random fields satisfying Markov Property) has been considered in Carmona and Fouque (1993).

The convergence result can also be proven in the case when $\alpha = 1$. In contrast with Eq. (2) the covariance matrix of the limiting Brownian motion is no longer given by an explicit formula and the technique used for proving convergence is quite different from the case of slowly oscillating fields. For more information on the subject a reader is asked to consult Papanicolaou and Varadhan (1982) and Osada (1982) for the case of time-independent and Fannjiang and Komorowski (1997) for time-dependent fields.

In our present article we apply a variant of the parametrix method for linear parabolic PDEs to prove weak convergence of the processes $\{X_\varepsilon(t)\}_{t\geqslant 0}$, as $\varepsilon\downarrow 0$ induced by Gaussian field $\boldsymbol{V}$ oscillating moderately in the spatial variable, i.e. the case when $0\leqslant \alpha < 1$. We shall prove that then the covariance matrix of the limiting Brownian motion is again given by Eq. (2). The idea of the proof can be summarized as follows. The key to prove tightness of the scaled trajectories is the control of the covariance matrix of the Lagrangian velocity of the particle $1/\varepsilon \boldsymbol{V}(t/\varepsilon^2, X_\varepsilon(t)/\varepsilon)$ when $\varepsilon\downarrow 0$. One can express the entries of this matrix using the random fundamental solution $p_\varepsilon(t,x,s,y)$ of the Kolmogorov's equation corresponding to the diffusion $\{X_\varepsilon(t)\}_{t\geqslant 0}$. We can expand the fundamental solution according to the classical parametrix expansion. The main observation made below is that in a time scale much larger than the microscopic scale ε^2 but still not too big, i.e. of order ε^γ, $\gamma<2$ the fundamental solution of the Kolmogorov's equation can be replaced, with a high degree of accuracy, by the fundamental solution of the corresponding equation whose coefficients are "frozen" in the spatial variable, i.e. the spatial argument is constant and equals the starting point of the diffusion path. The estimates of the covariance matrix of Lagrangian velocities can be made thanks to an explicit formula for such a fundamental solution. A non-rigorous argument of a similar nature has been applied also in the case when $\alpha = 1$ cf. Molchanov and Pitterbarg (1992). However in such a situation it is impossible to reduce the expansion to just a single term, instead it is necessary to work with an infinite series and this fact makes a rigorous proof by that method not available at the moment.

The organization of the paper is as follows. In Section 2 we shall introduce the notation and formulate the main result as well as a number of basic lemmas. In Section 3 we prove the main theorem using several lemmas which are crucial in obtaining both tightness and limit identification. We postpone their proofs until Section 5. In Section 4 we recall, without proofs (a reader is referred to Friedman (1963) for details), basic ideas of the parametrix expansion method.

2. Notation and the formulation of the main result

For any two vectors $a, b \in \mathbb{R}^d$ we denote by $a \otimes b$ the matrix $[a_i b_j]_{i,j=1\ldots d}$. We shall also write a^2 for $a \otimes a$. A scalar product of two tensors $\boldsymbol{A} = [a_{i_1,\ldots,i_k}]_{1 \leqslant i_1,\ldots,i_k \leqslant d}$ and $\boldsymbol{B} = [b_{i_1,\ldots,i_k}]_{1 \leqslant i_1,\ldots,i_k \leqslant d}$ is denoted by $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{i_1,\ldots,i_k=1}^{d} a_{i_1,\ldots,i_k} b_{i_1,\ldots,i_k}$.

Let $(\Omega, \mathscr{V}, P)$ be a certain probability space. By $\boldsymbol{E}$ we denote the expectation with respect to the probability measure P. Assume that on Ω we are given a group of measure preserving transformations $T_{t,x} : \Omega \to \Omega$, $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, i.e. $T_{s,x} T_{t,y} = T_{s+t,x+y}$, the mapping $(t,x,\omega) \mapsto T_{t,x}(\omega)$ is $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{V}$ to $\mathscr{V}$ measurable and $P[T_{t,x}(A)] = P[A]$ for any $A \in \mathscr{V}$, $(t,x) \in \mathbb{R} \times \mathbb{R}^d$. Here $\mathscr{B}_{\mathbb{R}^d}$ stands for the Borelian σ-algebra in $\mathbb{R}^d$.

Let $\tilde{\boldsymbol{V}} : \Omega \to \mathbb{R}^d$ be a certain random vector. $\boldsymbol{V}(t,x;\omega) = \tilde{\boldsymbol{V}}(T_{t,x}(\omega))$ defines then a d-dimensional stationary random field. We shall assume that it satisfies the following conditions.

C1 $\boldsymbol{V}(t,x)$ is a zero mean and Gaussian random field i.e. $\boldsymbol{E} V_i(0,0) = 0$, for $i = 1, \ldots, d$ and all its finite-dimensional distributions are Gaussian.

C2 The realisations of $\boldsymbol{V}$ are P a.s. continuous in t, of C^2 class in x and are incompressible i.e. $\mathrm{div}_x \boldsymbol{V}(t,x) \equiv \sum_{i=1}^{d} \partial_{x_i} V_i(t,x) = 0$.

Let us denote by $\mathscr{V}_{a,b}$ the σ-algebra generated by sets of the form $[\omega : \boldsymbol{V}(t,x;\omega) \in A]$, $a \leqslant t \leqslant b$, $x \in \mathbb{R}^d$ and $A \in \mathscr{B}_{\mathbb{R}^d}$. We shall define then the α-mixing coefficient for the field $\boldsymbol{V}$ (cf. [16]) by the formula

$$\alpha_{\mathscr{V}}(h) = \sup_{t} \sup_{A \in \mathscr{V}_{-\infty,t}, B \in \mathscr{V}_{t+h,+\infty}} |P(A \cap B) - P(A)P(B)|.$$

We suppose that

C3 for any $0 \leqslant \alpha < 1$ there exists $C(\alpha)$ such that

$$\alpha_{\mathscr{V}}(h) \leqslant \frac{C(\alpha)}{1 + h^{N(\alpha)}} \quad h \geqslant 0,$$

where

$$N(\alpha) = \frac{8}{2 - \alpha}.$$

Let us denote by $L^2_{a,b}$ the L^2-closure of the linear span formed over $V_i(t,x)$ $i=1,\dots,d$, $a\leqslant t\leqslant b$, $x\in\mathbb{R}^d$, where $(V_1(t,x),\dots,V_d(t,x))$ are the components of $\boldsymbol{V}(t,x)$. By $V_k^0(t,s,x)$ we denote the orthogonal projections onto $L^2_{-\infty,s}$ of $V_k(t,x)$, $k=1,\dots,d$. Let $V_k^1(t,s,x)$ be the orthogonal complement of $V_k(t,x)$ i.e. $V_k^1(t,s,x)=V_k(t,x)-V_k^0(t,s,x)$. We denote by $\mathscr{V}^{\perp}_{-\infty,s}$ the σ-algebra generated by $V_k^1(t,s,x)$, $t\geqslant s$, $x\in\mathbb{R}^d$, $k=1,\dots,d$. The following fact is a simple conclusion from a well-known theorem of Kolmogorov and Rozanov (see Rozanov, 1967) (p. 181 Theorems 10.1 and 10.2).

Lemma 1. *The σ-algebras $\mathscr{V}_{-\infty,s}$ and $\mathscr{V}^{\perp}_{-\infty,s}$ are independent.*

Let us denote by $\boldsymbol{R}(t,x)=\boldsymbol{E}[V_p(t,x)V_q(0,0)]$ the covariance matrix of the field $\boldsymbol{V}$. The following lemma is a direct consequence of Theorem 10.2 p. 181 Rozanov (1967) and Assumption C3.

Lemma 2. *There exists a constant $C>0$ depending only on α and $\boldsymbol{E}|\boldsymbol{V}(0,0)|^2$ such that*

$$\sum_{p,q=1}^{d}\sup_{x\in\mathbb{R}^d}|R_{p,q}(t,x)|\leqslant\frac{C}{1+t^{N(\alpha)}}.$$

and

$$\sum_{p=1}^{d}\|V_p^0(u,t,0)\|^2_{L^2}\leqslant\frac{C}{1+(u-t)^{N(\alpha)}}\quad \textit{for all } u\geqslant t.$$

From condition C2) we can also conclude the following.

Lemma 3. *There exists a constant $C>0$ depending only on $\boldsymbol{E}|\boldsymbol{V}(0,0)^2+\boldsymbol{E}|\nabla\boldsymbol{V}(0,0)|^2$ such that for all $x\in\mathbb{R}^d$*

$$\sum_{p=1}^{d}|R_{pp}(0,x)-R_{pp}(0,0)|\leqslant C|x|^2. \tag{3}$$

Here $\nabla\boldsymbol{V}(t,x)=[\partial_i V_j(t,x)]$.

This lemma follows from the fact that R_{pp}, $p=1,\dots,d$ are C^2 regular in x and all of their first partials vanish at 0 hence Eq. (3) holds in a neighborhood of 0. For large $|x|$ (3) is a consequence of boundedness of $|\boldsymbol{R}(0,x)|$, $x\in\mathbb{R}^d$.

Let $(\Sigma,\mathscr{L},Q)$ be a probability space equipped with a d-dimensional standard Brownian motion $\boldsymbol{B}(t)=(B_1(t),\dots,B_d(t))$, $t\geqslant 0$. By $\boldsymbol{M}$ we denote the expectation with respect to measure Q and by $\{\mathscr{L}_t\}_{t\geqslant 0}$ the history of the Brownian motion. In what follows we shall consider the family of processes $\{X_\varepsilon^{s,x}(t)\}_{t\geqslant s}$ defined on the probability space $(\Omega\times\Sigma,\ \mathscr{V}\otimes\mathscr{L},\ P\otimes Q)$ as follows:

$$X_\varepsilon^{s,x}(t)=x+\frac{1}{\varepsilon}\int_s^t \boldsymbol{U}_\varepsilon(\tau,X_\varepsilon^{s,x}(\tau))\,\mathrm{d}\tau+\sqrt{2}\boldsymbol{B}(t-s),\quad t\geqslant s, \tag{4}$$

where $\boldsymbol{U}_\varepsilon(t,x)=\boldsymbol{V}(t/\varepsilon^2,x/\varepsilon^\alpha)$. We denote also

$$\boldsymbol{U}_\varepsilon^1(t,s,x)=\boldsymbol{V}^1\left(\frac{t}{\varepsilon^2},\frac{s}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right) \quad \text{and} \quad \boldsymbol{U}_\varepsilon^0(t,s,x)=\boldsymbol{V}^0\left(\frac{t}{\varepsilon^2},\frac{s}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right). \tag{5}$$

In the case when $s=0$, $x=0$ we shall suppress the superscripts writing $X_\varepsilon^{s,x}(t)$. By $\tilde{\mathscr{V}}_t^\varepsilon$, $t\geqslant 0$ we denote the filtration of σ-algebras $\mathscr{V}_{-\infty,t/\varepsilon^2}\otimes\mathscr{X}_{t/\varepsilon^2}$, $t\geqslant 0$.

The following result is the main objective of this article.

Theorem 1. *Under the assumptions* C1–C3 *made about Gaussian field* $\boldsymbol{V}$ *the family of processes* $\{X_\varepsilon(t)\}_{t\geqslant 0}$, $\varepsilon>0$ *converges weakly over* $C([0,+\infty),\mathbb{R}^d)$, *as* $\varepsilon\downarrow 0$, *to a d-dimensional Brownian motion. The covariance matrix of the limiting Brownian motion* $\boldsymbol{A}=[a_{i,j}]$ *is given by*

$$a_{i,j}=\int_{-\infty}^{+\infty} R_{i,j}(t,0)\mathrm{d}t+2\delta_{i,j}, \quad i,j=1,\ldots,d. \tag{6}$$

Remark 1. The proof of Theorem 4 p. 121 of Kesten and Papanicolaou (1979) can be adapted to get a more direct condition, expressed in terms of the rate of decay of the spectral density matrix of the field, one needs to impose on the field in order to guarantee the required order of decay of the mixing coefficient in C3).

The following lemma is a direct consequence of Theorem 2 p. 500 of Port and Stone (1976).

Lemma 4. *Assume that* $\tilde{U}$ *is an integrable random variable on the probability space* $(\Omega\times\Sigma,\mathscr{V}\otimes\mathscr{X},P\otimes Q)$. *Then*

$$\boldsymbol{EM}\tilde{U}\circ\tau_t=\boldsymbol{EM}\tilde{U},$$

for any $t\geqslant 0$. *Here* $\tau_t:\Omega\times\Sigma\to\Omega\times\Sigma$ *is given by* $\tau_t(\omega,\xi)=(T^\varepsilon_{t,X_\varepsilon(t;\omega,\xi)}(\omega),\xi)$ *and* $T^\varepsilon_{t,x}=T_{t/\varepsilon^2,x/\varepsilon^\alpha}$.

3. The proof of the main result

In all our considerations throughout the remainder of this paper we shall assume that $\varepsilon\in(0,1]$. All the constants whose existence we shall claim throughout this section, unless stated otherwise, shall depend only on parameters $W=\boldsymbol{E}|\boldsymbol{V}(0,0)|^2+\boldsymbol{E}|\nabla\boldsymbol{V}(0,0)|^2+\boldsymbol{E}|\nabla\otimes\nabla\boldsymbol{V}(0,0)|^2$, α and T.

Proof of tightness. We shall prove that for arbitrary $T>0$ there exist constants $v,C>0$ such that

$$\boldsymbol{EM}\{|X_\varepsilon(t)-X_\varepsilon(s)|^2\Psi\}\leqslant C(t-s)\{\boldsymbol{EM}\Psi^v\}^{1/v} \tag{7}$$

holds for all Ψ non-negative and $\tilde{\mathscr{V}}_s^\varepsilon$-measurable, $0\leqslant s<t\leqslant T$ and $\varepsilon\in(0,1]$.

In light of the proof of Theorem 1 of Kesten and Papanicolaou (1979) (see, in particular, comments to formula (3.22) on p. 108) we can conclude from (7) that there exist $C, v_1, v_2, v_3 > 0$ such that

$$\boldsymbol{EM}\{|X_\varepsilon(u) - X_\varepsilon(t)|^{v_1}|X_\varepsilon(t) - X_\varepsilon(s)|^{v_2}\} \leqslant C(u-s)^{1+v_3} \tag{8}$$

for all $0 \leqslant s < t < u \leqslant T$. The family $\{X_\varepsilon(t)\}_{t\geqslant 0}$ is therefore tight in the Skorochod space $D([0,+\infty);\mathbb{R}^d)$. Since all relevant processes have continuous trajectories this suffices to claim tightness of the family over the space of continuous functions.

Let us partition $[s,t]$ with the points $t_k = s + k\varepsilon^\gamma$, $k = 0,\ldots,K$, $t_{k+1} = t$. The last interval $[t_K, t]$ is of length less than or equal to ε^γ, where the parameter $2 > \gamma > 0$ is to be specified later. We can write that

$$\boldsymbol{EM}[|X_\varepsilon(t) - X_\varepsilon(s)|^2\Psi] = \sum_{p=1}^{d}\sum_{k=0}^{K}\sum_{l=0}^{K} a_{\varepsilon,p,k,l}, \tag{9}$$

where $a_{\varepsilon,p,k,l} = \boldsymbol{EM}\{\Delta_{\varepsilon,p,k}\Delta_{\varepsilon,p,l}\Psi\}$ and $\Delta_{\varepsilon,p,k} = X_{\varepsilon,p}(t_{k+1}) - X_{\varepsilon,p}(t_k)$. $X_{\varepsilon,p}$, $p = 1,\ldots,d$ stand for the components of X_ε. We shall distinguish among three types of terms appearing on the right-hand side of (9), namely, those on the diagonal i.e. $k = 1$, close to the diagonal i.e. $l \neq k$ but $|l-k|$ is not too large and those far from the diagonal i.e. when $|l-k|$ is large.

In estimating the far off-diagonal terms we use the following lemma:

Lemma 5. *For an arbitrary $T > 0$ there exist a positive integer N, constants $C > 0$ and $\gamma_0 \in (0,2)$ such that for any $2 > \gamma \geqslant \gamma_0$ we can find $2 > \gamma' > \gamma$ for which the following holds*:

$$\sum_{p=1}^{d} |\boldsymbol{EM}\{\Delta X_{\varepsilon,p}(t')\Delta X_{\varepsilon,p}(t)\Psi\}| \leqslant C\varepsilon^{2\gamma'}(\boldsymbol{EM}\Psi^6)^{1/6}$$

for $T \geqslant t' \geqslant t + N\varepsilon^\gamma$ and any bounded, non-negative, $\tilde{\mathscr{V}}_t^\varepsilon$-measurable Ψ. Here $\Delta X_{\varepsilon,p}(t) = X_{\varepsilon,p}(t+\varepsilon^\gamma) - X_{\varepsilon,p}(t)$.

Remark 2. The introduction of the constant γ_0 in the statement of the lemma may seem superfluous at first. However, it will allow us later to choose a uniform partition length ε^γ according to various estimates we perform.

Let us choose γ_0 and N as in Lemma 5. For $l - k > N$ and any $\gamma \in [\gamma_0, 2)$ we can write that $|a_{\varepsilon,p,k,l}| \leqslant C\varepsilon^{2\gamma'}\{\boldsymbol{EM}\Psi^6\}^{1/6}$, if only $\gamma' > \gamma$ is chosen according to the previous lemma. In the case when $0 < l - k \leqslant N$ we estimate using the following lemma:

Lemma 6. *For any $T > 0$ and integer $K \geqslant 1$ there exist constants C, μ and $\gamma_0 \in (0,2)$ depending only on W, α, T and K such that for any $2 > \gamma \geqslant \gamma_0$ there is $\gamma' \in (\gamma, 2)$ for which the following holds*:

$$\sum_{p=1}^{d} |\boldsymbol{EM}\{\Delta X_{\varepsilon,p}(t + K\varepsilon^\gamma)\Delta X_{\varepsilon,p}(t)\Psi\}| \leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^\mu)^{1/\mu}, \quad 0 \leqslant t \leqslant T$$

for any bounded, non-negative, $\tilde{\mathscr{V}}_t^\varepsilon$-measurable Ψ.

Let us fix $K = N$. According to Lemma 6 we can find γ_0 such that for any $\gamma \in [\gamma_0, 2)$ there is $\gamma' > \gamma$ for which $|a_{\varepsilon,p,k,l}| \leqslant C\varepsilon^{2\gamma'}\{\boldsymbol{EM}\Psi^{\mu}\}^{1/\mu}$, provided that $0 < l - k \leqslant N$.

Finally, to estimate the terms with $l = k$ we shall need the following:

Lemma 7. *For any $T > 0$ there exist constants $\mu, C > 0$ and $2 > \gamma_0 > 0$ such that for any $2 > \gamma \geqslant \gamma_0$ and a bounded, non-negative, $\tilde{\mathscr{V}}_t^{\varepsilon}$-measurable Ψ the following estimate holds*:

$$\sum_{p=1}^{d} |\boldsymbol{EM}\{[X_{\varepsilon,p}(u) - X_{\varepsilon,p}(t)]^2\Psi\}| \leqslant C(u-t)(\boldsymbol{EM}\Psi^{\mu})^{1/\mu}$$

for $0 \leqslant t \leqslant u \leqslant t + \varepsilon^{\gamma} \leqslant T$.

Using Lemma 7 we get $|a_{\varepsilon,p,k,k}| \leqslant C(t_{k+1} - t_k)\{\boldsymbol{EM}\Psi^{\mu}\}^{1/\mu}$ provided that $\gamma \geqslant \gamma_0$ with γ_0 chosen according to Lemma 7.

Summarizing the above considerations we can conclude (7) from (9) upon a suitable choice of the constants C and v. This according to what we have stated at the beginning of this section suffices to conclude tightness.

Limit identification. To identify the limit we show that for any function $f \in C_0^{\infty}(\mathbb{R}^d)$ and a measure Q on $C([0,+\infty), \mathbb{R}^d)$ being the law of a possible weak limit of the family of processes $\{X_{\varepsilon}(t)\}_{t \geqslant 0}$ we have

$$f(x(t)) - \frac{1}{2}\sum_{p,q=1}^{d}\int_0^t a_{pq}\partial_{pq}^2 f(x(\varrho))\,\mathrm{d}\varrho, \quad x \in C([0,+\infty), \mathbb{R}^d) \tag{10}$$

is a martingale with respect to measure Q and the canonical filtration on the space of continuous functions. Using a well-known theorem on uniqueness of a solution of the martingale problem formulated in (10) (see e.g. Strook and Varadhan, 1979) we shall obtain uniqueness of the limiting measure for the given family of processes. In addition, we can identify it as a Wiener measure with the covariance matrix given by Eq. (6).

To prove that (10) is a martingale with respect to the limiting measure let us write

$$\begin{aligned}\boldsymbol{EM}\{[f(X_{\varepsilon}(t)) - f(X_{\varepsilon}(s))]\Psi\} &= \sum_{k=0}^{K}\boldsymbol{EM}\{[f(X_{\varepsilon}(t_{k+1})) - f(X_{\varepsilon}(t_k))]\Psi\}\\ &= \sum_{k=0}^{K}\left(A_k + \frac{1}{2}B_k + \frac{1}{6}C_k(\theta_k)\right).\end{aligned} \tag{11}$$

The last equality follows from Taylor expansion with $A_k = A_{k,k}$, $B_k = B_{k,k,k}$, $C_k(\theta) = C_{k,k,k}(\theta)$, where

$$\begin{aligned}A_{l,k} &= \boldsymbol{EM}[\Delta_{\varepsilon,l}\cdot\nabla f(X_{\varepsilon}(t_k))\Psi],\\ B_{m,l,k} &= \boldsymbol{EM}[\Delta_{\varepsilon,m}\otimes\Delta_{\varepsilon,l}\cdot\nabla\otimes\nabla f(X_{\varepsilon}(t_k))\Psi],\\ C_{m,l,k}(\theta) &= \boldsymbol{EM}[\Delta_{\varepsilon,m}\otimes\Delta_{\varepsilon,l}\otimes\Delta_{\varepsilon,k}\cdot\nabla\otimes\nabla\otimes\nabla f(\theta)\Psi].\end{aligned} \tag{12}$$

θ_k-s of Eq. (11) are chosen suitably on the segments $[X_\varepsilon(t_k), X_\varepsilon(t_{k+1})]$ and $\Delta_{\varepsilon,k}$ is the vector with components $\Delta_{\varepsilon,p,k}$, $p=1,\ldots,d$. $C_k(\theta)$ can be estimated then by

$$\frac{1}{6}\sup_x |\nabla\otimes\nabla\otimes\nabla f(x)| \boldsymbol{EM}\left|\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} \boldsymbol{V}\left(\frac{\varrho}{\varepsilon^2}, \frac{X_\varepsilon(\varrho)}{\varepsilon^\alpha}\right) \mathrm{d}\varrho\right|^3 \sup \Psi. \tag{13}$$

Thanks to Lemma 4 we can conclude that $|C_k(\theta)| \leqslant C\varepsilon^{3\gamma-3}$.

Using again the Taylor expansion of ∇f around $X_\varepsilon(t_{k-N})$ we get that

$$A_{k,k} = A_{k,k-N} + \sum_{l=1}^{N} B_{k,k-l,k-N} + \frac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{N} C_{k,k-l,k-m}(\theta_k^*), \tag{14}$$

where N is fixed and shall be specified later. Here $\theta_k^* \in [X_\varepsilon(t_{k-N}), X_\varepsilon(t_k)]$. Arguing as before we can see that the last term of Eq. (14) is of order $O(\varepsilon^{3\gamma-3})$. To estimate $A_{k,k-N}$ we need the following:

Lemma 8. *For an arbitrary $T>0$ there exist a positive integer N, constants $C>0$ and $\gamma_0 \in (0,2)$ such that for any $2>\gamma\geqslant\gamma_0$ there exists $2>\gamma'>\gamma$ for which the following holds*

$$\sum_{p=1}^{d} |\boldsymbol{EM}\{\Delta X_{\varepsilon,p}(t')\Psi\}| \leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^{3/2})^{2/3}$$

for any $t' \geqslant t + N\varepsilon^\gamma$ and arbitrary Ψ bounded, non-negative and $\tilde{\mathscr{V}}_t^\varepsilon$- measurable.

Taking N, γ_0 as in the statement of the lemma we can write that for arbitrary $2>\gamma\geqslant\gamma_0$ one can choose $\gamma' \in (\gamma, 2)$ such that $|A_{k,k-N}| \leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^{3/2})^{2/3}$. $|B_{k,k-l,k-N}|$ can be estimated by $C\varepsilon^{\gamma'}$ with the help of Lemma 6 with K chosen as an element of $\{1,\ldots,N\}$. Summarizing we obtain that $|A_k| \leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^\mu)^{1/\mu}$ for certain constants $C>0$, $2>\gamma'>\gamma$. To estimate B_k-s appearing in Eq. (11) we shall need the following.

Lemma 9. *For an arbitrary $T>0$ there exist a positive integer N, constants $\mu, C>0$ and $\gamma_0 \in (0,2)$ such that for any $2>\gamma\geqslant\gamma_0$ we can find $2>\gamma'>\gamma$ for which the following holds*:

$$|\boldsymbol{EM}\{[\Delta X_{\varepsilon,p}(t')\Delta X_{\varepsilon,q}(t') - a_{pq}\varepsilon^\gamma]\Psi\}| \leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^\mu)^{1/\mu}, \quad p,q=1,\ldots,d$$

for any $t' \geqslant t+N\varepsilon^\gamma$ and bounded, non-negative, $\tilde{\mathscr{V}}_t^\varepsilon$-measurable Ψ. Here a_{pq} for $p,q=1,\ldots,d$ are given by Eq. (6).

Using this lemma we get, via a similar argument to the one applied before that

$$|B_k - \tfrac{1}{2}\boldsymbol{A}\cdot\nabla\otimes\nabla f(X_\varepsilon(t_k))(t_{k+1}-t_k)\Psi| \leqslant C\varepsilon^{\gamma'} \wedge (t_{k+1}-t_k)(\boldsymbol{EM}\Psi^\mu)^{1/\mu},$$

for certain $C, \mu > 0$. Summarizing we obtain

$$\lim_{\varepsilon\to 0} \boldsymbol{EM}\{[f(X_\varepsilon(t)) - f(X_\varepsilon(s)) - \frac{1}{2}\sum_{p,q=1}^{d}\int_s^t a_{pq}\partial_{pq}^2 f(X_\varepsilon(\varrho))\,\mathrm{d}\varrho]\Psi\} = 0.$$

Hence, the law of any limiting measure satisfies the martingale problem (10), which ends the proof of the theorem. □

4. An outline of the parametrix method

Before taking up the task of proving the lemmas stated in the previous section, let us briefly recall the parametrix method for linear parabolic P.D.Es which is the key ingredient of the proofs of those lemmas. All formulas we present here can be proven rigorously in a precisely the same way as it is done in Friedman (1963).

Let $p(t,x,s,y)$ be the *fundamental solution* of (15) i.e. it satisfies formally the following Cauchy problem:

$$\begin{aligned} &\partial_t p(t,x,s,y) = \Delta_x p(t,x,s,y) + \boldsymbol{b}(t,x)\cdot\nabla_x p(t,x,s,y), \quad t \geqslant s, \\ &p(s,x,s,y) = \delta(x-y), \end{aligned} \tag{15}$$

where $\boldsymbol{b}=(b_1,\ldots,b_d)$ is a certain d-dimensional, locally Hölderian vector field and δ denotes Dirac's distribution. Let us set

$$q(t,x,s,y) = q\left(t-s, x-y-\int_s^t \boldsymbol{b}(\tau,y)\,\mathrm{d}\tau\right), \tag{16}$$

$$\boldsymbol{Q}(t,x,s,y) = \nabla_x q(t,x,s,y),$$

where $q(t,x) = (1/4\pi t)^{d/2}\exp\{-|x|^2/t\}$. Then formally

$$L^y_{t,x} q(t,x,s,y) = \delta(t-s)\otimes\delta(x-y),$$

where

$$L^y_{t,x} = \partial_t - \Delta_x - \boldsymbol{b}(t,y)\cdot\nabla_x. \tag{17}$$

We can write therefore that

$$L_{t,x} q(t,x,s,y) = (\boldsymbol{b}(t,x)-\boldsymbol{b}(t,y))\cdot\nabla_x q(t,x,s,y) + \delta(t-s)\otimes\delta(x-y). \tag{18}$$

Here, the operator $L_{t,x}$ is defined by Eq. (17) with $\boldsymbol{b}(t,y)$ replaced by $\boldsymbol{b}(t,x)$. Applying formally the operator $(L_{t,x})^{-1}$ to both sides of Eq. (18) and using the fact that $(L_{t,x})^{-1}(\delta(t-s)\otimes\delta(x-y)) = p(t,x,s,y)$ we get, after a simple computation, that

$$p(t,x,s,y) = q(t,x,s,y) + \int_s^t \int_{\mathbb{R}^d} p(t,x,\tau,z)(\boldsymbol{b}(\tau,y)-\boldsymbol{b}(\tau,z))\cdot\nabla_z q(\tau,z,s,y)\,\mathrm{d}\tau\,\mathrm{d}z.$$

Iterating M times we get

$$p(t,x,s,y) = \sum_{m=0}^{M} q_m(t,x,s,y) + r_M(t,x,s,y), \tag{19}$$

for any integer $M \geqslant 0$. Here

$$q_m(t,x,s,y) = \int\cdots\int_{s\leqslant\tau_m\leqslant\cdots\leqslant\tau_1\leqslant t} \int_{(\mathbb{R}^d)^m} q(t,x,\tau_1,z_1)\prod_{l=1}^{m} Z(\tau_l,\tau_{l+1},z_l,z_{l+1})\,\mathrm{d}\tau^{(m)}\,\mathrm{d}z^{(m)},$$

where $\mathrm{d}z^{(m)} = \mathrm{d}z_1 \cdots \mathrm{d}z_m$, $\mathrm{d}\tau^{(m)} = \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_m$. The indexed terms with no integrations performed we denote $\tau_{m+1} = s$, $z_{m+1} = y$. In addition,

$$Z(t,s,x,y) = \boldsymbol{T}(t,x,y) \cdot \boldsymbol{Q}(t,x,s,y), \qquad \boldsymbol{T}(t,x,y) = \boldsymbol{b}(t,y) - \boldsymbol{b}(t,x).$$

The remainder term is equal to

$$r_M(t,x,s,y) = \int_{s \leqslant \tau_{M+1} \leqslant \cdots \leqslant \tau_1 \leqslant t} \cdots \int \int_{(\mathbb{R}^d)^{M+1}} p(t,x,\tau_1,z_1) \times \prod_{l=1}^{M+1} Z(\tau_l,\tau_{l+1},z_l,z_{l+1}) \, \mathrm{d}\tau^{(M+1)} \, \mathrm{d}z^{(M+1)}. \tag{20}$$

5. The proofs of the basic lemmas

Throughout this section we suppress the subscript ε of X and the superscript ε of $\tilde{\mathscr{V}}$. By $T_{t,x}$ we denote $T_{t/\varepsilon^2, x/\varepsilon^\alpha}$. C shall stand for any generic constant independent of ε.

5.1. The proof of Lemma 7

From Eq. (4) we get

$$X_p(u) - X_p(t) = \sqrt{2}[B_p(u) - B_p(t)] + \frac{1}{\varepsilon} \int_t^u U_{\varepsilon,p}(\tau, X(\tau)) \, \mathrm{d}\tau. \tag{21}$$

Squaring both sides of Eq. (21), multiplying by Ψ and applying expectations $\boldsymbol{E}, \boldsymbol{M}$ we get

$$\begin{aligned} \boldsymbol{EM}\{[X_p(u) - X_p(t)]^2 \Psi\} &= \frac{2\sqrt{2}}{\varepsilon} \int_t^u \boldsymbol{EM}\{[B_p(\tau) - B_p(t)] U_{\varepsilon,p}(\tau, X(\tau)) \Psi\} \, \mathrm{d}\tau \\ &\quad + \frac{2}{\varepsilon^2} \int_t^u \mathrm{d}\tau \int_t^\tau \boldsymbol{EM}[U_{\varepsilon,p}(\tau, X(\tau)) U_{\varepsilon,p}(\tau', X(\tau')) \Psi] \, \mathrm{d}\tau' \\ &\quad + 2\boldsymbol{EM}\{[B_p(u) - B_p(t)]^2 \Psi\}. \end{aligned} \tag{22}$$

The last term equals $2(u-t)\boldsymbol{EM}\Psi$. We shall estimate the first and second terms which for a brevity sake will be denoted by J_1 and J_2, respectively.

The estimation of J_1. Notice that

$$J_1 = \frac{2\sqrt{2}}{\varepsilon} \int_t^u \boldsymbol{EM}\{[B_p(\tau) - B_p(t)][U_{\varepsilon,p}(\tau, X(\tau)) - U_{\varepsilon,p}(\tau, X(t))] \Psi\} \, \mathrm{d}\tau. \tag{23}$$

Applying Itô's formula to the process $U_{\varepsilon,p}(\tau,X(\varrho))$, $\varrho\geqslant t$ we get that $J_1=J_{11}+J_{12}+J_{13}$, where

$$J_{11}=\frac{4}{\varepsilon^{1+\alpha}}\int_t^u \boldsymbol{EM}\{[B_p(\tau)-B_p(t)]\int_t^\tau \boldsymbol{W}_{\varepsilon,p}(\tau,X(\varrho))\,\mathrm{d}\boldsymbol{B}(\varrho)\Psi\}\,\mathrm{d}\tau, \tag{24}$$

$$J_{12}=\frac{2\sqrt{2}}{\varepsilon^{2+\alpha}}\int_t^u \mathrm{d}\tau\int_t^\tau \boldsymbol{EM}\{\boldsymbol{W}_{\varepsilon,p}(\tau,X(\varrho))\cdot\boldsymbol{U}_\varepsilon(\varrho,X(\varrho))[B_p(\tau)-B_p(t)]\Psi\}\,\mathrm{d}\varrho,$$

$$J_{13}=\frac{2\sqrt{2}}{\varepsilon^{1+2\alpha}}\int_t^u \mathrm{d}\tau\int_t^\tau \boldsymbol{EM}\{L_{\varepsilon,p}(\tau,X(\varrho))[B_p(\tau)-B_p(t)]\Psi\}\,\mathrm{d}\varrho.$$

Here $\boldsymbol{W}_{\varepsilon,p}(t,x)=\nabla_x V_p(t/\varepsilon^2,x/\varepsilon^\alpha)$, $L_{\varepsilon,p}(t,x)=\Delta_x V_p(t/\varepsilon^2,x/\varepsilon^\alpha)$. Using Hölder inequality we can write that

$$|J_{11}|\leqslant\frac{4}{\varepsilon^{1+\alpha}}(\boldsymbol{EM}\Psi^4)^{1/4}\times\int_t^u\left\{\boldsymbol{EM}\left[\int_t^\tau \boldsymbol{W}_{\varepsilon,p}(\tau,X(\varrho))\cdot\mathrm{d}\boldsymbol{B}(\varrho)\right]^2\right\}^{1/2}\{\boldsymbol{M}[B_p(\tau)-B_p(t)]^4\}^{1/4}\,\mathrm{d}\tau. \tag{25}$$

Since $u-t<\varepsilon^\gamma$, applying Lemma 4, we get

$$|J_{11}|\leqslant C(u-t)\varepsilon^{\gamma-1-\alpha}\{\boldsymbol{E}|\nabla V_p(0,0)|^2\}^{1/2}(\boldsymbol{EM}\Psi^4)^{1/4}.$$

Choosing $\gamma_0>1+\alpha$ we obtain the desired bound.

The estimates of J_{12},J_{13} are done in a complete analogy with those made in (25) and are based on applications of Hölder inequality together with Lemma 4. As a result we get

$$|J_{12}|\leqslant C(u-t)\varepsilon^{3/2\gamma-2-\alpha}(\boldsymbol{EM}\Psi^4)^{1/4}\quad\text{and}\quad|J_{13}|\leqslant C(u-t)\varepsilon^{3/2\gamma-1-2\alpha}(\boldsymbol{EM}\Psi^4)^{1/4}.$$

The constant C depends on absolute moments of the field, its gradient and laplacian. Choosing γ_0 such that $3/2\gamma_0-2-\alpha>0$ and $3/2\gamma_0-1-2\alpha>0$ we get the desired bound.

Estimation of J_2. Suppose that $f:\mathbb{R}^d\to\mathbb{R}$ is a continuous function, for which $|f(y)|\leqslant C\mathrm{e}^{C|y|^{2-\delta}}$ where $\delta,C>0$. Let $u(s,y)=\boldsymbol{M}f(X^{s,y}(t))$ for $t\geqslant 0$. Here $X^{s,y}$ is the trajectory given by Eq. (4). As it is well known (see e.g. Strook and Varadhan (1979)) u satisfies the following Cauchy problem for a backward parabolic PDE:

$$\begin{aligned}&\partial_s u(s,y;\omega)+\Delta_y u(s,y;\omega)+\tfrac{1}{\varepsilon}\boldsymbol{U}_\varepsilon(s,y;\omega)\cdot\nabla_y u(s,y;\omega)=0,\quad t\geqslant s,\\&u(t,y;\omega)=f(y).\end{aligned} \tag{26}$$

Thanks to the fact that $\mathrm{div}_y\,\boldsymbol{U}_\varepsilon(s,y;\omega)=0$ we can write

$$u(s,y;\omega)=\int p_\varepsilon(t,x,s,y;\omega)f(x)\,\mathrm{d}x, \tag{27}$$

where $p_\varepsilon(t,x,s,y;\omega)$, $t \geqslant s, x, y \in \mathbb{R}^d$ is the fundamental solution of Eq. (15) with the drift $\boldsymbol{b}(t,x)$ equal to $-\frac{1}{\varepsilon}\boldsymbol{U}_\varepsilon(t,x;\omega)$. Stationarity of $\boldsymbol{U}_\varepsilon$ and uniqueness of solutions of Eq. (27) together imply that

$$p_\varepsilon(t,x,s,y;T_{u,z}(\omega)) = p_\varepsilon(t+u,x+z,s+u,y+z;\omega). \tag{28}$$

The following lemma is closely related to Lemma 4 and enables us to rewrite the expression for J_2 in a form more suitable for our subsequent computations.

Lemma 10. *Suppose that $\tau \geqslant \tau' \geqslant 0$ and a random variable $\Psi : \Omega \times \Sigma \to \mathbb{R}$ is $\tilde{\mathscr{V}}_{\tau'}$-measurable. Then there exists a $\mathscr{V}_{-\infty,0} \otimes \mathscr{Z}_{\tau'/\varepsilon^2}$-measurable $\tilde{\Psi} : \Omega \times \Sigma \to \mathbb{R}$ such that the following are true*:
(1)

$$\boldsymbol{EM}[f(T_{\tau,X(\tau;\omega,\xi)}(\omega))\Psi(\omega,\xi)] = \boldsymbol{EM}[f(\omega)\tilde{\Psi}(\omega,\xi)] \tag{29}$$

for any bounded random variable $f : \Omega \to \mathbb{R}$.
(2) $\|\tilde{\Psi}\|_{L^p} \leqslant \|\Psi\|_{L^p}$ *for all* $p \geqslant 1$.

Proof. Suppose first that $0 < \tau' < \tau$. Using Markov Property of diffusions and Eq. (27) one can write that

$$\boldsymbol{M}[f(T_{\tau,X(\tau;\omega,\xi)}(\omega))\Psi(\omega,\xi)] = \boldsymbol{M}[f(T_{\tau,X^{\tau',X(\tau')}(\tau;\omega,\xi)}(\omega))\Psi(\omega,\xi)]$$
$$= \boldsymbol{M}\left[\int_{\mathbb{R}^d} f(T_{\tau,y}(\omega))p_\varepsilon(\tau,y,\tau',X(\tau');\omega)\Psi(\omega,\xi)\,\mathrm{d}y\right]. \tag{30}$$

P invariance of $T_{t,y}$ and Eq. (28) together imply that the left-hand side of Eq. (29) equals

$$\boldsymbol{EM}\left[f(\omega)\int_{\mathbb{R}^d} p_\varepsilon(\tau,y,\tau',X(\tau';T_{-\tau,-y}(\omega),\xi);T_{-\tau,-y}(\omega))\Psi(T_{-\tau,-y}(\omega),\xi)\,\mathrm{d}y\right]$$
$$= \boldsymbol{EM}[f(\omega)\tilde{\Psi}(\omega,\xi)], \tag{31}$$

where

$$\tilde{\Psi} = \int_{\mathbb{R}^d} p_\varepsilon(0,0,\tau'-\tau,X^{-\tau,-y}(\tau'-\tau;\omega,\xi);\omega)\Psi(T_{-\tau,-y}(\omega),\xi)\,\mathrm{d}y.$$

The last equality in Eq. (31) follows from Eq. (28) and the fact that $X(\tau';T_{-\tau,-y}(\omega),\xi) - y = X^{-\tau,-y}(\tau'-\tau;\omega,\xi)$. $\tilde{\Psi}$ is $\mathscr{V}_{-\infty,0} \otimes \mathscr{Z}_{\tau'/\varepsilon^2}$-measurable, thanks to measurability of the mappings $(y,\omega,\xi) \mapsto X^{-\tau,-y}(\tau'-\tau;\omega,\xi)$ and $(y,\omega,\xi) \mapsto p_\varepsilon(0,0,\tau'-\tau,y;\omega)$ with respect to σ-algebra $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{V}_{-\infty,0} \otimes \mathscr{Z}_{\tau'/\varepsilon^2}$.

As for (2) it holds for $p = \infty$. Using part (1) of the lemma for $f \equiv 1$ we get (2) when $p = 1$. The convexity property of the norm allows us to extend (2) for an arbitrary $p \geqslant 1$. Using an approximation procedure we can generalize the above proof to the case when $\tau' = \tau$ as well. $\square$

An application of the Markov Property of diffusions gives

$$J_2=\frac{2}{\varepsilon^2}\int_t^u \mathrm{d}\tau\int_t^\tau \boldsymbol{EM}\{\boldsymbol{M}[U_{\varepsilon,p}(\tau,X^{\tau',X(\tau')}(\tau;\omega,\xi'))]V_p(\tau',X(\tau'))\Psi\}\,\mathrm{d}\tau'$$
$$=\frac{2}{\varepsilon^2}\int_t^u \mathrm{d}\tau\int_t^\tau \boldsymbol{EM}\{\boldsymbol{M}[U_{\varepsilon,p}(\tau-\tau',X(\tau-\tau';T_{\tau',X(\tau')}(\omega),\xi');T_{\tau',X(\tau')}(\omega))]$$
$$\times V_p(0,0;T_{\tau',X(\tau')}(\omega))\Psi\}\,\mathrm{d}\tau'.$$

Here the latter expectation $\boldsymbol{M}$ is computed in ξ' variable. Using Lemma 10 and changing variables $\tau:=\tau-t,\ \tau':=\tau-\tau'$ we arrive at

$$J_2=\frac{2}{\varepsilon^2}\int_0^{u-t} \mathrm{d}\tau\int_0^\tau \boldsymbol{EM}\{\boldsymbol{M}[U_{\varepsilon,p}(\tau',X(\tau'))]V_p(0,0)\Psi_{\tau,\tau'}\}\,\mathrm{d}\tau', \qquad (32)$$

where $\Psi_{\tau,\tau'}(\omega,\xi)$ is $\mathscr{V}_{-\infty,0}\otimes\mathscr{Z}_{t/\varepsilon^2}$-measurable and such that $\boldsymbol{EM}\Psi^p_{\tau,\tau'}\leqslant\boldsymbol{EM}\Psi^p$ for all $p\geqslant 1$. Using Eq. (27) we can write that

$$\boldsymbol{M}U_{\varepsilon,p}(\tau',X(\tau'))=\int_{\mathbb{R}^d} p_\varepsilon(\tau',x,0,0)U_{\varepsilon,p}(\tau',x)\,\mathrm{d}x.$$

We shall expand p_ε using the parametrix method presented in Section 4 in the context of our present setting. We can write then that $J_2=\sum_{m=0}^M Q_m+R_M$. Here

$$Q_m=\frac{2}{\varepsilon^2}\int_0^{u-t}\mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\int_{\mathbb{R}^d}\boldsymbol{EM}[\mathscr{Q}_m(\tau',x,0,0)U_{\varepsilon,p}(\tau',x)V_p(0,0)\Psi_{\tau,\tau'}]\,\mathrm{d}x, \qquad (33)$$

$$\mathscr{Q}_m(t,x,s,y)=\frac{1}{\varepsilon^m}\underset{s\leqslant\tau_m\leqslant\cdots\leqslant\tau_1\leqslant t}{\int\cdots\int}\int_{(\mathbb{R}^d)^m} q_\varepsilon(t,x,\tau_1,z_1)$$
$$\times\prod_{l=1}^m Z_\varepsilon(\tau_l,\tau_{l+1},z_l,z_{l+1})\,\mathrm{d}z^{(m)}\,\mathrm{d}\tau^{(m)}$$

and $\mathrm{d}z^{(m)}=\mathrm{d}z_1\ldots\mathrm{d}z_m$, $\mathrm{d}\tau^{(m)}=\mathrm{d}\tau_1\ldots\mathrm{d}\tau_m$. The indexed terms with no integration performed denote correspondingly s and y.

$$R_M=\frac{2}{\varepsilon^2}\int_0^{u-t}\mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\int_{\mathbb{R}^d}\boldsymbol{EM}[\mathscr{R}_M(\tau',x,0,0)U_{\varepsilon,p}(\tau',x)V_p(0,0)\Psi_{\tau,\tau'}]\,\mathrm{d}x, \qquad (34)$$

where

$$\mathscr{R}_M(t,x,s,y)=\frac{1}{\varepsilon^{1+M}}\underset{s\leqslant\tau_{M+1}\leqslant\cdots\leqslant\tau_1\leqslant t}{\int\cdots\int}\int_{(\mathbb{R}^d)^{M+1}} p_\varepsilon(t,x,\tau_1,z_1)$$
$$\times\prod_{l=1}^{M+1} Z_\varepsilon(\tau_l,\tau_{l+1},z_l,z_{l+1})\,\mathrm{d}\tau^{(M+1)}\,\mathrm{d}z^{(M+1)}.$$

In analogy with Eq. (19) we have set

$$\boldsymbol{T}_\varepsilon(t,x,y) = \boldsymbol{U}_\varepsilon(t,x) - \boldsymbol{U}_\varepsilon(t,y), \tag{35}$$

$$Z_\varepsilon(t,s,x,y) = \boldsymbol{T}_\varepsilon(t,x,y) \cdot \boldsymbol{Q}_\varepsilon(t,x,s,y), \tag{36}$$

q_ε and $\boldsymbol{Q}_\varepsilon$ are given by Eq. (16) where the drift $\boldsymbol{b}(t,x)$ is replaced by $-1/\varepsilon \boldsymbol{U}_\varepsilon(t,x)$.

Remark 3. In the sequel we shall need a bit more general notation concerning the parametrix expansion. For a fixed $0 \leqslant \theta \leqslant 1$ and u let us denote by $p_\varepsilon^{\theta,u}$ the fundamental solution of the Cauchy problem (26) with the drift $-1/\varepsilon[\boldsymbol{U}_\varepsilon^1(t,u,x) + \theta \boldsymbol{U}_\varepsilon^0(t,u,x)]$ (cf. (5)). Let us observe that $p_\varepsilon^{1,u} = p_\varepsilon$. The terms obtained by expanding $p_\varepsilon^{\theta,u}$ shall be denoted by $\mathscr{Q}_m^{\theta,u}(t,x,s,y)$, $\mathscr{R}_M^{\theta,u}(t,x,s,y)$ in correspondence with $\mathscr{Q}_m$ and $\mathscr{R}_M$ defined before. In the definition of these expressions we use $q_\varepsilon^{\theta,u}(t,x,s,y)$, $\boldsymbol{Q}_\varepsilon^{\theta,u}(t,x,s,y)$, $\boldsymbol{T}_\varepsilon^{\theta,u}(t,x,y)$ and $Z_\varepsilon^{\theta,u}(s,t,x,y)$ defined via formulas (16) and (35)–(36), in which we replace $\boldsymbol{U}_\varepsilon(t,x)$ by $\boldsymbol{U}_\varepsilon^1(t,u,x) + \theta \boldsymbol{U}_\varepsilon^0(t,u,x)$.

Estimates of R_M. The following estimates of the heat kernel hold.

Lemma 11. *There exists a constant C depending only on d such that*

$$|\boldsymbol{Q}(t,x)| \leqslant \frac{C}{\sqrt{t}} q(2t,x) \quad \textit{and} \quad |\Delta(t,x)| \leqslant \frac{C}{t} q(2t,x).$$

Here $\boldsymbol{Q}(t,x) = \nabla_x q(t,x)$.

Proof. Let $C = 2^{d/2-1} \sup_{t \geqslant 0} t\mathrm{e}^{-t^2/2}$. We have

$$|\boldsymbol{Q}(t,x)| = 2^{d/2-1} \frac{|x|}{t} \mathrm{e}^{-|x|^2/2t} q(2t,x) \leqslant \frac{C}{\sqrt{t}} q(2t,x). \qquad \square$$

Using Lemma 11 we can write that

$$\boldsymbol{Q}_\varepsilon(\tau_k, z_k, \tau_{k+1}, z_{k+1}; 1) \leqslant \frac{C}{\sqrt{\tau_k - \tau_{k+1}}} q_\varepsilon(\tau_k, z_k, \tau_{k+1}, z_{k+1}; 2).$$

Here

$$q_\varepsilon(t,x,s,y;\sigma) = q(\sigma(t-s), x - y + \frac{1}{\varepsilon} \int_s^t \boldsymbol{U}_\varepsilon(\tau', y)\,\mathrm{d}\tau')$$

and $\boldsymbol{Q}_\varepsilon(\cdot,\cdot,\cdot,\cdot;\sigma)$ is defined as its gradient in x variable. We get then that

$$\begin{aligned}
|R_M| \leqslant{}& \frac{C}{\varepsilon^{3+M}} \int_{0 \leqslant \tau_{M+1} \leqslant \cdots \leqslant \tau \leqslant t-u} \cdots \int \int_{(\mathbb{R}^d)^{M+2}} \boldsymbol{EM}\{ p_\varepsilon(\tau', x, \tau_1, z_1) \\
&\times \prod_{k=1}^{M+1} \left[\frac{1}{\sqrt{\tau_k - \tau_{k+1}}} |\boldsymbol{T}_\varepsilon(\tau_k, z_k, z_{k+1})| q_\varepsilon(\tau_k, z_k, \tau_{k+1}, z_{k+1}; 2) \right] \\
&\times |U_{\varepsilon,p}(\tau', x) V_p(0,0)| \Psi_{\tau,\tau'} \}\, \mathrm{d}\hat{\tau}^{(M+1)}\, \mathrm{d}z^{(M+1)}\, \mathrm{d}x,
\end{aligned} \tag{37}$$

where by convention $z_{M+2}=0$, $\tau_{M+2}=0$, $\mathrm{d}\hat{\tau}^{(M+1)}=\mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_{M+1}$. Performing the integration in x variable we get

$$\int_{\mathbb{R}^d} p_\varepsilon(\tau',x,\tau_1,z_1)|U_{\varepsilon,p}(\tau',x)|\,\mathrm{d}x=\boldsymbol{M}|U_{\varepsilon,p}(\tau',X^{\tau_1,z_1}(\tau'))|. \tag{38}$$

Using Eq. (38) and Hölder inequality we get an estimate as follows:

$$|R_M|\leqslant\frac{CW}{\varepsilon^{3+M}}(\boldsymbol{EM}\Psi^2)^{1/2}\int\cdots\int_{0\leqslant\tau_{M+1}\leqslant\cdots\leqslant\tau\leqslant u-t}\int_{(\mathbb{R}^d)^{M+1}}$$

$$\prod_{k=1}^{M+1}\left\{\frac{T_{k,10(M-1)}}{\sqrt{\tau_k-\tau_{k+1}}}\left[\boldsymbol{E}q_\varepsilon\left(\tau_k,z_k,\tau_{k+1},z_{k+1};\frac{1}{5(M+1)}\right)\right]^{1/(10(M+1))}\right\}\mathrm{d}z^{(M+1)}.$$

Here $T_{k,m}=[\boldsymbol{E}|\boldsymbol{T}_\varepsilon(\tau_k,z_k,z_{k+1})|^m]^{1/m}$ and

$$W=[\boldsymbol{EM}|U_{\varepsilon,p}(\tau',X^{\tau_1,z_1}(\tau'))|^{20}]^{1/20}=(\boldsymbol{E}V_p(0,0)^{20})^{1/20},$$

thanks to Lemma 4.

Using elementary properties of Gaussian variables we can conclude that there exists a constant C depending on m such that

$$T_{k,m}=C\sum_{p=1}^{d}\left[R_{pp}(0,0)-R_{pp}\left(0,\frac{z_k-z_{k+1}}{\varepsilon^\alpha}\right)\right]^{1/2}.$$

Due to Lemma 3 we can estimate the last expression by $C/\varepsilon^\alpha|z_k-z_{k+1}|$.

On the other hand for arbitrary $c_1,c_2>0$

$$\begin{aligned}&\{\boldsymbol{E}[q_\varepsilon(\tau_k,z_k,\tau_{k+1},z_{k+1};c_2)]^{c_1}\}^{1/c_1}\\&=C\left(\frac{1}{\tau_k-\tau_{k+1}}\right)^{d/2}\left\{[c_1c_2(\tau_k-\tau_{k+1})]^{d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\lambda\cdot(z_k-z_{k+1})}\boldsymbol{E}\right.\\&\left.\times\exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_{\tau_{k+1}}^{\tau_k}\lambda\cdot\boldsymbol{U}_\varepsilon(\varrho,z_{k+1})\,\mathrm{d}\varrho-\frac{c_1c_2|\lambda|^2}{2}(\tau_k-\tau_{k+1})\right\}\mathrm{d}\lambda\right\}^{1/c_1}.\end{aligned} \tag{39}$$

Since $\boldsymbol{E}\mathrm{e}^{\mathrm{i}\eta}=\mathrm{e}^{-1/2\boldsymbol{E}\eta^2}$, for any Gaussian variable η, we can write that the left-hand side of Eq. (39) equals

$$\begin{aligned}&C\left(\frac{1}{\tau_k-\tau_{k+1}}\right)^{d/2}\left\{[c_1c_2(\tau_k-\tau_{k+1})]^{d/2}\int\exp\left\{\mathrm{i}\lambda\cdot(z_k-z_{k+1})\right.\right.\\&\left.\left.-\frac{1}{2}\int_0^{\tau_k-\tau_{k+1}}\mathrm{d}\varrho\int_0^{\varrho/\varepsilon^2}\boldsymbol{R}_S(\varrho',0)\cdot\lambda^2\,\mathrm{d}\varrho'-\frac{c_1c_2|\lambda|^2}{2}(\tau_k-\tau_{k+1})\right\}\mathrm{d}\lambda\right\}^{1/c_1},\end{aligned} \tag{40}$$

where $\boldsymbol{R}_S(t,x)=\boldsymbol{R}(t,x)+\boldsymbol{R}^{\mathrm{T}}(t,x)$, $\lambda^2=\lambda\otimes\lambda$ (cf. Section 2).

The left-hand side of Eq. (39) can be therefore written as being equal to

$$C\left(\frac{1}{\tau_k-\tau_{k+1}}\right)^{d/2}\left(\frac{c_1c_2}{\det\boldsymbol{D}}\right)^{1/c_1}\exp\left\{-\frac{\boldsymbol{D}^{-1}\cdot(z_k-z_{k+1})^2}{c_1(\tau_k-\tau_{k+1})}\right\},$$

where

$$\boldsymbol{D}=c_1c_2\boldsymbol{I}+\frac{1}{\tau_k-\tau_{k+1}}\int_0^{\tau_k-\tau_{k+1}}\mathrm{d}\varrho\int_0^{\varrho/\varepsilon^2}\boldsymbol{R}_S(\varrho',0)\,\mathrm{d}\varrho'\leqslant\Lambda\boldsymbol{I},$$

for some sufficiently large Λ. Hence, in conclusion

$$\{\boldsymbol{E}[q_\varepsilon(\tau_k,z_k,\tau_{k+1},z_{k+1};c_2)]^{c_1}\}^{1/c_1}\leqslant Cq(\Lambda(\tau_k-\tau_{k+1}),z_k-z_{k+1}), \tag{41}$$

for a certain constant $C>0$ depending only on c_1,c_2.

Applying Eq. (41) for $c_1=10(M+1)$, $c_2=2$ we can bound $|R_M|$ from above as follows:

$$|R_M|\leqslant\frac{C(\boldsymbol{E}\boldsymbol{M}\Psi^2)^{1/2}}{\varepsilon^{3+M+(M+1)\alpha}}\int\cdots\int_{0\leqslant\tau_{M+1}\leqslant\cdots\leqslant\tau\leqslant u-t}\int_{(\mathbb{R}^d)^{M+1}}$$

$$\times\prod_{k=1}^{M+1}\left[\frac{|z_k-z_{k+1}|}{\sqrt{\tau_k-\tau_{k+1}}}q(\Lambda(\tau_k-\tau_{k+1}),z_k-z_{k+1})\right]\mathrm{d}\hat{\tau}^{(M+1)}\,\mathrm{d}z^{(M+1)}.$$

Using Lemma 11 and we get that

$$|\boldsymbol{R}_M|\leqslant\frac{C(\boldsymbol{E}\boldsymbol{M}\Psi^2)^{1/2}}{\varepsilon^{3+M+(M+1)\alpha}}\int\cdots\int_{0\leqslant\tau_{M+1}\leqslant\cdots\leqslant\tau\leqslant u-t}\int_{(\mathbb{R}^d)^{M+1}}$$

$$\times\prod_{k=1}^{M+1}q(2\Lambda(\tau_k-\tau_{k+1}),z_k-z_{k+1})\,\mathrm{d}\hat{\tau}^{(M+1)}\,\mathrm{d}z^{(M+1)}. \tag{42}$$

Performing integrations in z variables we end up with the following estimate:

$$|\boldsymbol{R}_M|\leqslant\frac{C}{\varepsilon^{3+M+(M+1)\alpha}}(u-t)^{M+3}(\boldsymbol{E}\boldsymbol{M}\Psi^2)^{1/2}\leqslant C(u-t)\varepsilon^{\delta}(\boldsymbol{E}\boldsymbol{M}\Psi^2)^{1/2}, \tag{43}$$

for $\delta=(\gamma-1-\alpha)(M+1)+\gamma-2>0$, if only $\gamma_0>1+\alpha$ and $M>(2-\gamma_0)/(\gamma_0-1-\alpha)$. We have used here the fact that $u-t\leqslant\varepsilon^{\gamma}$.

Remark 4. Precisely, the same argument as described above gives the following estimate for $\mathscr{R}_M^{\theta,u}$ for arbitrary $N\geqslant 1$:

$$[\boldsymbol{E}[\mathscr{R}_M^{\theta,u}(t,x,s,y)]^N]^{1/N}\leqslant\frac{C(t-s)^{M+1}}{\varepsilon^{(M+1)(\alpha+1)}}q(\Lambda(t-s),x-y).$$

Here C is independent of $0\leqslant\theta\leqslant 1$, $0\leqslant u\leqslant T$ and $\Lambda>0$ is sufficiently large.

Estimates of Q_m, $0<m\leqslant M$. These estimates are similar to those just done for $\boldsymbol{R}_M$ with a significant simplification due to the fact that the expression for Q_m in Eq. (33) does not involve an unknown p_ε, instead all terms defining Q_m are given explicitly. We just briefly outline the argument which a reader should be able to recover using the relevant parts of computations for $\boldsymbol{R}_M$.

After estimating $|Z_\varepsilon|$ in formula (33) with the help of Lemma 11 we obtain the following:

$$|Q_m| \leqslant \frac{C}{\varepsilon^{2+m}} \int_{0\leqslant\tau_m\leqslant\cdots\leqslant\tau\leqslant u-t}\cdots\int \int_{(\mathbb{R}^d)^{m+1}} \boldsymbol{EM}\left\{\prod_{k=1}^{m}\left[\frac{1}{\sqrt{\tau_k-\tau_{k+1}}}|\boldsymbol{T}_\varepsilon(\tau_k,z_k,z_{k+1})|\right]\right.$$
$$\left.\times\prod_{k=0}^{m} q_\varepsilon(\tau_k,z_k,\tau_{k+1},z_{k+1};2)|U_{\varepsilon,p}(\tau',x)V_p(0,0)|\Psi_{\tau,\tau'}\right\}\mathrm{d}\hat{\tau}^{(m)}\,\mathrm{d}z^{(m)}\,\mathrm{d}x.$$

Here, by convention $\tau_0=\tau'$, $z_0=x$, $\tau_{m+1}=0$, $z_{m+1}=0$.

Applying Hölder inequality to estimate the average $\boldsymbol{EM}$ and estimating along the same lines as we have done in the case of $\boldsymbol{R}_M$ we get

$$|Q_m| \leqslant \frac{C(\boldsymbol{EM}\Psi^2)^{1/2}}{\varepsilon^{2+m+m\alpha}} \int_{0\leqslant\tau_m\leqslant\cdots\leqslant\tau\leqslant u-t}\cdots\int \int_{(\mathbb{R}^d)^{m+1}}$$
$$\times\prod_{k=0}^{m} q(\Lambda(\tau_k-\tau_{k+1}),z_k-z_{k+1})\,\mathrm{d}\hat{\tau}^{(m)}\,\mathrm{d}z^{(m)}\,\mathrm{d}x,$$

which, again, by virtue of Lemma 11 can be bounded from above by

$$|Q_m|\leqslant\frac{C(u-t)^{m+2}}{\varepsilon^{m+2+m\alpha}}(\boldsymbol{EM}\Psi^2)^{1/2}\leqslant C(u-t)\varepsilon^{\delta}(\boldsymbol{EM}\Psi^2)^{1/2}, \tag{44}$$

for $\delta=\gamma(m+1)-m-2-m\alpha>0$, if only $\gamma_0>(3+\alpha)/2$. This method breaks down in the case when $m=0$. We shall separate the relevant computations and present them in the next paragraph.

Remark 5. We have indeed proven the following estimate of $\mathscr{Q}_m^{\theta,u}$ for arbitrary $N\geqslant 1$

$$[\boldsymbol{E}[\mathscr{Q}_m^{\theta,u}(t,x,s,y)]^N]^{1/N}\leqslant\frac{C(t-s)^m}{\varepsilon^{m(1+\alpha)}}q(\Lambda(t-s),x-y).$$

C is again independent of $0\leqslant\theta\leqslant 1$, $0\leqslant u$, ϱ, $\leqslant T$ and $\Lambda>0$ is sufficiently large.

Estimates of Q_0. Using Eq. (33) we can write that

$$Q_0=\frac{C}{\varepsilon^2}\int_0^{u-t}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\tau'\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\lambda\cdot x}$$

$$\boldsymbol{EM}\left[\exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_0^{\tau'}\boldsymbol{U}_\varepsilon(\varrho,x)\cdot\lambda\,\mathrm{d}\varrho-\tau'|\lambda|^2\right\}U_{\varepsilon,p}(\tau',x)V_p(0,0)\Psi_{\tau,\tau'}\right]\mathrm{d}x\,\mathrm{d}\lambda,$$

where C is some constant depending only on d. Using Lemma 1 we can write $\boldsymbol{U}_\varepsilon=\boldsymbol{U}_\varepsilon^0+\boldsymbol{U}_\varepsilon^1$ (cf. (5)) where $\boldsymbol{U}_\varepsilon^0$ and $\boldsymbol{U}_\varepsilon^1$ are, respectively, $\mathscr{V}_{-\infty,0}$ and $\mathscr{V}_{-\infty,0}^{\perp}$-measurable.

We have then $Q_0 = Q_{0,0} + Q_{0,1}$, where

$$Q_{0,j} = \frac{C}{\varepsilon^2}\int_0^{u-t} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\tau' \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\lambda\cdot x} \boldsymbol{EM}\left[\exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_0^{\tau'} \boldsymbol{U}_\varepsilon^1(\varrho,0,x)\cdot\lambda\,\mathrm{d}\varrho - |\lambda|^2\tau'\right\}\right.$$
$$\left.\times\exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_0^{\tau'} \boldsymbol{U}_\varepsilon^0(\varrho,0,x)\cdot\lambda\,\mathrm{d}\varrho\right\} U^j_{\varepsilon,p}(\tau',0,x)V_p(0,0)\Psi_{\tau,\tau'}\right]\mathrm{d}x\,\mathrm{d}\lambda, \quad j=0,1.$$

We shall deal with these two terms separately.

Case $j=0$. Since $\int_0^{\tau'} \boldsymbol{U}_\varepsilon^1(\varrho,0,x)\cdot\lambda\,\mathrm{d}\varrho$ is independent of the σ-algebra $\mathscr{V}_{-\infty,0}$ we can write that (cf. Eq. (39))

$$Q_{0,0} = \frac{C}{\varepsilon^2}\int_0^{u-t} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\tau' \int_{\mathbb{R}^d} \frac{1}{\sqrt{\det(\tau'\boldsymbol{G})}} \boldsymbol{EM}[F(x,x)U^0_{\varepsilon,p}(\tau',0,x)V_p(0,0)\Psi_{\tau,\tau'}]\,\mathrm{d}x.$$

Here

$$F(x,y) = \exp\left\{-\frac{1}{2}\left(y + \frac{1}{\varepsilon}\int_0^{\tau'} \boldsymbol{U}_\varepsilon^0(\varrho,0,x)\,\mathrm{d}\varrho\right)^2 \cdot (\tau'\boldsymbol{G})^{-1}\right\}$$

and $\boldsymbol{G}$ is a certain symmetric matrix independent of x,y but possibly dependent on τ' satisfying

$$2\boldsymbol{I} \leqslant \boldsymbol{G} \leqslant \Lambda\boldsymbol{I}, \tag{45}$$

for Λ sufficiently large. Thanks to Eq. (45) we can estimate $|Q_{0,0}|$ by

$$|Q_{0,0}| \leqslant \frac{C}{\varepsilon^2}\int_0^{u-t} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\tau'$$
$$\times \int_{\mathbb{R}^d} \boldsymbol{EM}\{q_\varepsilon^{(0)}(\tau',x,0,0;\Lambda)|U^0_{\varepsilon,p}(\tau',0,x)V_p(0,0)|\Psi_{\tau,\tau'}\}\,\mathrm{d}x. \tag{46}$$

Here $q_\varepsilon^{(0)}$ is defined by Eq. (16) where the drift $\boldsymbol{b}(t,x)$ is replaced by $-1/\varepsilon\boldsymbol{U}_\varepsilon^0(t,0,x)$. Applying Hölder inequality we obtain

$$|Q_{0,0}| \leqslant \frac{C}{\varepsilon^2}(\boldsymbol{EM}\Psi^6)^{1/6}\int_0^{u-t} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\tau' \int_{\mathbb{R}^d} \|V_p^0(\tau',0,0)\|_{L^2}$$
$$\times \{\boldsymbol{E}[q_\varepsilon^{(0)}(\tau',x,0,0;\Lambda)]^6\}^{1/6}\,\mathrm{d}x.$$

Estimating the sixth moment of $q_\varepsilon^{(0)}$ as in Eq. (39) we get, by virtue of Lemma 2 that

$$|Q_{0,0}| \leqslant C(\boldsymbol{EM}\Psi^6)^{1/6}\int_0^{u-t} \mathrm{d}\tau \int_0^{\tau/\varepsilon^2} \frac{\mathrm{d}\tau'}{1+(\tau')^2} \leqslant C(u-t)(\boldsymbol{EM}\Psi^6)^{1/6}. \tag{47}$$

Case $j=1$. For any Gaussian vector (η,ζ) we have $\boldsymbol{E}\eta e^{i\zeta} = i\boldsymbol{E}(\eta\zeta)e^{-1/2\boldsymbol{E}\zeta^2}$. We can therefore write that

$$Q_{0,1} = \frac{C}{\varepsilon^3}\sum_{q=1}^{d}\int_0^{u-t} d\tau \int_0^{\tau} d\tau' \int_0^{\tau'} d\varrho \int_{\mathbb{R}^d} \left\{ R_{pq}^{1,1}\left(\frac{\tau'}{\varepsilon^2},\frac{\varrho}{\varepsilon^2},0\right)\frac{1}{\sqrt{\det(\tau'\boldsymbol{G})}} \right.$$

$$\left. \times \boldsymbol{EM}\{\partial_y F(x,y)|_{y=x} V_p(0,0)\Psi_{\tau,\tau'}\} \right\} dx. \tag{48}$$

Here $R_{pq}^{1,1}(\tau,\varrho,x) = \boldsymbol{E}\{V_p^1(\tau,0,x)V_q^1(\varrho,0,0)\}$, $\boldsymbol{G}$ is a certain positive-definite symmetric matrix such that there exists a sufficiently large Λ for which (45) holds.

Notice that

$$\partial_y F(x,y)|_{y=x} = \partial_x[F(x,x)] - \partial_x F(x,y)|_{y=x}. \tag{49}$$

The latter term on the right-hand side of Eq. (49) is equal to

$$-\frac{1}{\varepsilon^{1+\alpha}\tau'}\boldsymbol{G}^{-1}\cdot\left[\left(x+\frac{1}{\varepsilon}\int_0^{\tau'}\boldsymbol{U}_\varepsilon^0(\varrho,0,x)\,d\varrho\right)\otimes\int_0^{\tau'}\Gamma_q^0\left(\frac{\varrho}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)d\varrho\right]$$

$$\times\exp\left\{-\frac{1}{2}(\tau'\boldsymbol{G})^{-1}\cdot\left(x+\frac{1}{\varepsilon}\int_0^{\tau'}\boldsymbol{U}_\varepsilon^0(\varrho,0,x)\,d\varrho\right)^2\right\}. \tag{50}$$

Here $\Gamma_q^0(t,x)$ is the projection of $\partial_{x_q}\boldsymbol{V}(t,x)$ onto $L^2_{-\infty,0}$. Substituting the right-hand side of Eq. (49) into Eq. (48) we get that the integral over dx of the term corresponding to the first term on the right-hand side of Eq. (49) vanishes. Applying Hölder inequality to the expression arising from the term given by Eq. (50) we shall have the following bound on $Q_{0,1}$.

$$|Q_{0,1}| \leqslant \frac{C}{\varepsilon^{4+\alpha}}(\boldsymbol{EM}\Psi^2)^{1/2}\sum_{q=1}^{d}\int_0^{u-t}d\tau\int_0^{\tau}d\tau'\int_0^{\tau'}d\varrho\int_{\mathbb{R}^d}\sqrt{\tau'}R_{pq}^{1,1}\left(\frac{\tau'}{\varepsilon^2},\frac{\varrho}{\varepsilon^2},0\right)$$

$$\times\left\{\boldsymbol{E}\left[\frac{|x+1/\varepsilon\int_0^{\tau'}\boldsymbol{U}_\varepsilon^0(\varrho,0,x)\,d\varrho|}{\sqrt{\tau'}}q_\varepsilon^{(0)}(\tau',x,0,0;\Lambda)\right]^6\right\}^{1/6}dx.$$

Using Lemma 11 we can write that

$$|Q_{0,1}| \leqslant \frac{CR}{\varepsilon^{2+\alpha}}(\boldsymbol{EM}\Psi^2)^{1/2}\sum_{q=1}^{d}\int_0^{u-t}d\tau\int_0^{\tau}\sqrt{\tau'}$$

$$\times\int_{\mathbb{R}^d}\{\boldsymbol{E}[q_\varepsilon^{(0)}(\tau',x,0,0;2\Lambda)]^6\}^{1/6}\,d\tau'\,dx.$$

Here $R=\sup_{\varrho>0}\int_0^{\varrho}R_{pq}^{1,1}(\varrho,\varrho',0)\,d\varrho' < +\infty$. Estimating $\{\boldsymbol{E}[q_\varepsilon^{(0)}(\tau',x,0,0;2\Lambda)]^6\}^{1/6}$ as in (39) we get that $|Q_{0,1}| \leqslant C(u-t)\varepsilon^{3/2\gamma-2-\alpha}(\boldsymbol{EM}\Psi^2)^{1/2}$, which guarantees the desired estimate provided that $\frac{3}{2}\gamma_0-2-\alpha>0$. This concludes the proof of the lemma. □

5.2. The proof of Lemma 6

Let us assume that $\gamma \geqslant \gamma_0$ where γ_0 is given in the proof of Lemma 7. Repeating the calculations leading to Eq. (22) we get that

$$\boldsymbol{EM}\{\Delta X_p(t+K\varepsilon^\gamma)\Delta X_p(t)\Psi\} = K_1 + K_2, \tag{51}$$

where

$$K_1 = \frac{\sqrt{2}}{\varepsilon}\int_{t+K\varepsilon^\gamma}^{t+(K+1)\varepsilon^\gamma} \boldsymbol{EM}\{[U_{\varepsilon,p}(\tau,X(\tau)) - U_{\varepsilon,p}(\tau,X(t))] \times [B_p(t+\varepsilon^\gamma) - B_p(t)]\Psi\}\,\mathrm{d}\tau,$$

$$K_2 = \frac{1}{\varepsilon^2}\int_{t+K\varepsilon^\gamma}^{t+(K+1)\varepsilon^\gamma} \mathrm{d}\tau \int_t^{t+\varepsilon^\gamma} \boldsymbol{EM}[U_{\varepsilon,p}(\tau,X(\tau))U_{\varepsilon,p}(\tau',X(\tau'))\Psi]\,\mathrm{d}\tau'.$$

The argument using Itô's formula and Hölder inequality applied to estimate (23) yields in this case that $K_1 \leqslant C\varepsilon^{\gamma'}$ for some choice of C, $\gamma' > \gamma$, where $\gamma \geqslant \gamma_0$. Let us remark here that the choice of C may depend on K.

We shall estimate K_2 in a similar fashion as we have estimated J_2 in the previous proof. First, we can write, in analogy with Eq. (32), that

$$K_2 = \frac{1}{\varepsilon^2}\int_{N\varepsilon^\gamma}^{(N+1)\varepsilon^\gamma} \mathrm{d}\tau \int_{\tau-\varepsilon^\gamma}^{\tau} \boldsymbol{EM}\{\boldsymbol{M}[U_{\varepsilon,p}(\tau',X(\tau'))]V_p(0,0)\Psi_{\tau,\tau'}\}\,\mathrm{d}\tau',$$

where $\Psi_{\tau,\tau'}$ is a certain $\mathscr{V}_{-\infty,0}\otimes\mathscr{Z}_{(\tau-\tau')/\varepsilon^2}$-measurable and such that $\boldsymbol{E}\Psi_{\tau,\tau'}^p \leqslant \boldsymbol{E}\Psi^p$ for all $p \geqslant 1$.

An application of the parametrix method enables us to write that $K_2 = \sum_{m=0}^M Q'_m + R'_M$, where Q'_m, R'_M are given as in formulas (33) and (34) with the obvious adjustment of the first two integrals in the multiple integration used in respective definitions. The terms Q'_m for $0 < m \leqslant M$ and R'_M are estimated in the same fashion as their counterparts representing J_2. We can conclude that for any $\gamma \geqslant \gamma_0$ there exists $\gamma' > \gamma$ and C possibly depending on K such that $|R'_M| \leqslant C\varepsilon^{\gamma'}$. Likewise $|Q'_m| \leqslant C\varepsilon^{\gamma'}$, for $0 < m \leqslant M$.

Mimicking relevant calculations from the previous proof we can write that $Q'_0 = Q'_{0,0} + Q'_{0,1}$, where

$$Q'_{0,j} = \frac{C}{\varepsilon^2}\int_{K\varepsilon^\gamma}^{(K+1)\varepsilon^\gamma} \mathrm{d}\tau \int_{\tau-\varepsilon^\gamma}^{\tau} \mathrm{d}\tau' \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\lambda\cdot x} \times \boldsymbol{EM}\left[\exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_0^{\tau'} \boldsymbol{U}_\varepsilon^1(\varrho,0,x)\cdot\lambda\,\mathrm{d}\varrho - |\lambda|^2\tau'\right\} \times \exp\left\{\frac{\mathrm{i}}{\varepsilon}\int_0^{\tau'} \boldsymbol{U}_\varepsilon^0(\varrho,0,x)\cdot\lambda\,\mathrm{d}\varrho\right\} U_{\varepsilon,p}^j(\tau',0,x)V_p(0,0)\Psi_{\tau,\tau'}\right]\mathrm{d}x\,\mathrm{d}\lambda, \quad j=0,1.$$

The estimates of $Q'_{0,j}$-s can be repeated line by line from the resepective part of the argument contained in the previous proof. The term corresponding to $j=0$ is then estimated by (cf. (47)) $|Q'_{0,0}| \leqslant C\varepsilon^2$ while $|Q'_{0,1}| \leqslant C\varepsilon^{5/2\gamma-2-\alpha}$ and the conclusion of the lemma holds. $\square$

5.3. The proofs of Lemmas 5, 8 and 9

Let r be any positive integer and $p_1,\dots,p_r\in\{1,\dots,d\}$, $t_1,\dots,t_r\in\mathbb{R}$ and $y\in\mathbb{R}^d$. Denote

$$W_{p_1\cdots p_r}(t_1,\dots,t_r,y)=U_{\varepsilon,p_1}(t_1,y)\cdots U_{\varepsilon,p_r}(t_r,y), \tag{52}$$

$$\tilde W_{p_1\cdots p_r}(t_1,\dots,t_r,y)=W_{p_1\cdots p_r}(t_1,\dots,t_r,y)-\boldsymbol{E}W_{p_1\cdots p_r}(t_1,\dots,t_r,0). \tag{53}$$

We shall also introduce $W^{1,u}_{p_1\cdots p_r}$ and $\tilde W^{1,u}_{p_1\cdots p_r}$, which are given by formulas (52) and (53) provided that we replace $U_{\varepsilon,p}(t,x)$ by $U^1_{\varepsilon,p}(t,u,x)$ in the relevant product.

The proofs of Lemmas 5, 8 and 9 shall be established as corollaries of the following lemma:

Lemma 12. *Let $r,p_1,\dots,p_r$, be as specified above. There exist a positive integer N, constants $C>0$ and $2>\gamma_0>0$ such that for an arbitrary $2>\gamma\geqslant\gamma_0$ we can find $2>\gamma'>\gamma$, for which the following holds*:

$$|\boldsymbol{EM}[\tilde W_{p_1\cdots p_r}(t_1,\dots,t_r,X(t_1))\Psi]|\leqslant C\varepsilon^{2\gamma'}(\boldsymbol{EM}\,\Psi^{3/2})^{2/3} \tag{54}$$

for any $t_r\geqslant\cdots\geqslant t_1\geqslant t+N\varepsilon^\gamma$ and a bounded, non-negative, $\tilde{\mathscr V}_t$-measurable Ψ.

Proof. Suppose that $N>5/(1-\gamma/2)$, where $2>\gamma>\gamma_0$. γ_0 is, for now, chosen as in the proof of Lemma 7. We shall modify it further during the course of the proof. Let us denote by I the expression whose absolute value we estimate in (54). Using Markov Property of diffusions together with Lemma 10 we get

$$I=\boldsymbol{EM}\{\boldsymbol{M}[\tilde W_{p_1\cdots p_r}(N\varepsilon^\gamma,t_2',\dots,t_r',X(N\varepsilon^\gamma))]\tilde\Psi\},$$

where $t_2'=t_2-t_1+N\varepsilon^\gamma,\dots,t_r'=t_r-t_1+N\varepsilon^\gamma$, $\tilde\Psi$ is $\mathscr V_{-\infty,0}\otimes\mathscr Z_{t/\varepsilon^2}$-measurable and $\boldsymbol{E}\tilde\Psi^p\leqslant\boldsymbol{E}\Psi^p$ for all $p\geqslant 1$. In the sequel we shall omit writing the primes by t_k-s. Using Eq. (27) the above can be further written as

$$\begin{aligned}I&=\int\boldsymbol{ME}[p_\varepsilon(N\varepsilon^\gamma,x,0,0)\tilde W_{p_1\cdots p_r}(N\varepsilon^\gamma,t_2,\dots,t_r,x)\tilde\Psi]\,\mathrm{d}x\\&=\int_{(\mathbb{R}^d)^N}\boldsymbol{ME}[p_\varepsilon(\varrho_N,w_N,\varrho_{N-1},w_{N-1})\cdots p_\varepsilon(\varrho_1,w_1,0,0)\\&\quad\times\tilde W_{p_1\cdots p_r}(\varrho_N,t_2,\dots,t_r,w_N)\tilde\Psi]\,\mathrm{d}w^{(N)},\end{aligned}\tag{55}$$

where $\varrho_k=k\varepsilon^\gamma$, $k=1,\dots,N$, $\mathrm{d}w^{(N)}=\mathrm{d}w_1\cdots\mathrm{d}w_N$.

Expanding each factor p_ε, as described in Section 4, we can write that $I=I_1+I_2$, where

$$\begin{aligned}I_1=\sum_{S,T}\sum_{m_{j_1}\cdots m_{j_{N-q}}=0}^{M}\int_{(\mathbb{R}^d)^N}\boldsymbol{ME}\Bigg[&\prod_{l=1}^{q}\mathscr R_M(\varrho_{i_l},w_{i_l},\varrho_{i_l-1},w_{i_l-1})\\&\times\prod_{l=1}^{N-q}\mathscr Q_{m_{j_l}}(\varrho_{j_l},w_{j_l},\varrho_{j_l-1},w_{j_l-1})\tilde W_{p_1\cdots p_r}(\varrho_N,t_2,\dots,t_r,w_N)\tilde\Psi\Bigg]\,\mathrm{d}w^{(N)}.\end{aligned}\tag{56}$$

Here $\varrho_0 = 0$, $w_0 = 0$ and the summation $\sum_{S,T}$ extends over all disjoint subsets $S = \{i_1, \ldots, i_q\}$, $T = \{j_1, \ldots, j_{N-q}\}$ of $\{1, \ldots, N\}$ such that set S is non-empty.

$$I_2 = \sum_{m_1 \cdots m_N = 0}^{M} \int_{(\mathbb{R}^d)^N} \boldsymbol{ME} \left[\prod_{l=1}^{N} \mathscr{Q}_{m_l}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) \right. \\ \left. \times \tilde{W}_{p_1 \cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N) \tilde{\Psi} \right] \mathrm{d}w^{(N)}.$$

Factors $\mathscr{Q}_m, \mathscr{R}_m$ have been defined in Eqs. (33) and (34) correspondingly. Using Remarks 4 and 5 we shall obtain that

$$|I_1| \leqslant C\varepsilon^{2\gamma} \sum_{S,T} \sum_{m_{j_1} \cdots m_{j_{N-q}} = 0}^{M} \varepsilon^{\delta} (\boldsymbol{EM} \tilde{\Psi}^{3/2})^{2/3},$$

where $\delta = q(\gamma - 1 - \alpha)(M + 1) + \sum_{r=1}^{N-q} m_{j_r}(\gamma - 1 - \alpha) - 2\gamma$ and S, T are as in Eq. (56). Choosing $\gamma_0 > 1 + \alpha$ and $M > 4/(\gamma - 1 - \alpha)$ we shall get the desired bound.

Next, we write $I_2 = I_{21} + I_{22}$. Here

$$I_{21} = \sum_{k=1}^{N} \sum_{m_1 \cdots m_N = 0}^{M} \int_{(\mathbb{R}^d)^N} \boldsymbol{ME} \left\{ \prod_{l=k+1}^{N} \Delta \mathscr{Q}_{m_l} \mathscr{Q}_{m_k}^{0, \varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}) \right. \\ \left. \times \prod_{l=1}^{k-1} \mathscr{Q}_{m_l}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) \tilde{W}_{p_1 \cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N) \tilde{\Psi} \right\} \mathrm{d}w^{(N)}, \tag{57}$$

where $\mathscr{Q}_{m_k}^{\theta, \varrho_{k-1}}$ is defined in Remark 3,

$$\Delta \mathscr{Q}_{m_l} = \mathscr{Q}_{m_l}^{1, \varrho_{l-1}}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) - \mathscr{Q}_{m_l}^{0, \varrho_{l-1}}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) \tag{58}$$

and

$$I_{22} = \sum_{m_1 \cdots m_N = 0}^{M} \int_{(\mathbb{R}^d)^N} \int_{[0,1]^N} \boldsymbol{ME} \left[\prod_{l=1}^{N} \frac{\mathrm{d}}{\mathrm{d}\theta_l} \mathscr{Q}_{m_l}^{\theta_l, \varrho_{l-1}}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) \right. \\ \left. \times \tilde{W}_{p_1 \cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N) \tilde{\Psi} \right] \mathrm{d}w^{(N)} \, \mathrm{d}\theta^{(N)},$$

with $\mathrm{d}\theta^{(N)} = \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N$.

Estimates of I_{22}. Using Hölder inequality we can write that

$$|I_{22}| \leqslant \prod_{i=1}^{r} (\boldsymbol{E} V_{p_i}(0,0)^{6r})^{1/6r} \sum_{m_1 \cdots m_N = 0}^{M} \int_{(\mathbb{R}^d)^N} \int_{[0,1]^N} \prod_{k:\, m_k > 0} \{\boldsymbol{E}[\Delta \mathscr{Q}_{m_k}]^{6N}\}^{1/6N} \\ \times \prod_{k: m_k = 0} \left\{ \boldsymbol{E} \left[\frac{\mathrm{d}}{\mathrm{d}\theta_k} \mathscr{Q}_0^{\theta_k, \varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}) \right]^{6N} \right\}^{1/6N} \mathrm{d}w^{(N)} \, \mathrm{d}\hat{\theta}^{(N)} \\ \times (\boldsymbol{EM} \tilde{\Psi}^{3/2})^{2/3}.$$

Here $\mathrm{d}\hat{\theta}^{(N)} = \prod_{k:m_k=0} \mathrm{d}\theta_k$. Since $\varrho_k - \varrho_{k-1} = \varepsilon^\gamma$, by virtue of Remark 5 we have

$$\{\boldsymbol{E}[\Delta \mathscr{Q}_{m_k}]^{6N}\}^{1/6N} \leqslant C\varepsilon^{(\gamma-1-\alpha)m_k} q(\Lambda\varepsilon^\gamma, w_k - w_{k-1}) \tag{59}$$

for some sufficiently large $\Lambda > 0$.

On the other hand $\mathrm{d}/\mathrm{d}\theta_k \mathscr{Q}_0^{\theta_k,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1})$ is equal to

$$\frac{1}{\varepsilon}\int_{\varrho_{k-1}}^{\varrho_k} \boldsymbol{U}_\varepsilon^0(\varrho, \varrho_{k-1}, w_{k-1}) \cdot \boldsymbol{Q}_\varepsilon^{\theta,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1})\,\mathrm{d}\varrho.$$

We can write therefore that

$$\begin{aligned}&\left\{\boldsymbol{E}\left[\frac{\mathrm{d}}{\mathrm{d}\theta_k}\mathscr{Q}_0^{\theta_k,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1})\right]^{6N}\right\}^{1/6N}\\ &\quad\leqslant \{\boldsymbol{E}[\boldsymbol{Q}_\varepsilon^{\theta,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1})]^{12N}\}^{1/12N}\\ &\quad\times\frac{1}{\varepsilon}\int_{\varrho_{k-1}}^{\varrho_k}\{\boldsymbol{E}[\boldsymbol{U}_\varepsilon^0(\varrho, \varrho_{k-1}, w_{k-1})]^{12N}\}^{1/12N}\,\mathrm{d}\varrho.\end{aligned} \tag{60}$$

From the heat kernel estimates of Lemma 11 we can conclude that the first factor on the right-hand side of (60) is bounded from above by

$$\begin{aligned}&\frac{C}{\sqrt{\varrho_k - \varrho_{k-1}}}\{\boldsymbol{E}[q_\varepsilon^{\theta,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}; 2)]^{12N}\}^{1/12N}\\ &\quad\leqslant \frac{C}{\sqrt{\varrho_k - \varrho_{k-1}}} q(\Lambda(\varrho_k - \varrho_{k-1}), w_k - w_{k-1}).\end{aligned}$$

for some sufficiently large $\Lambda > 0$. The last inequality can be obtained by repeating the argument leading to (41).

Using elementary properties of Gaussian variables we get that the expression under the integral on the right-hand side of (60) equals $C_N\{\boldsymbol{E}[\boldsymbol{U}_\varepsilon^0(\varrho, \varrho_{k-1}, w_{k-1})]^2\}^{1/2}$. Remembering that $\varrho_k - \varrho_{k-1} = \varepsilon^\gamma$ we can conclude, by virtue of Lemma 2, that the left-hand side of (60) is less than or equal to

$$\frac{C}{\varepsilon^{1+\gamma/2}} q(\Lambda\varepsilon^\gamma, w_{k+1} - w_k)\int_{\varrho_{k-1}}^{\varrho_k}\frac{\mathrm{d}\varrho}{1 + ((\varrho - \varrho_{k-1})/\varepsilon^2)^2} \leqslant C\varepsilon^{1-\gamma/2} q(\Lambda\varepsilon^\gamma, w_{k+1} - w_k).$$

Finally, we get that $|I_{22}| \leqslant C\varepsilon^\delta(\boldsymbol{EM}\tilde{\Psi}^{3/2})^{2/3}$, where $\delta \geqslant N(1-\gamma/2) + L(3\gamma/2 - 2 - \alpha)$. L stands for the cardinality of the set of positive m_k-s. Choosing $\gamma_0 > \frac{2}{3}(2+\alpha)$ and remembering the choice of N we have made at the beginning of the proof we obtain the desired estimate.

Estimates of I_{21}. Let us observe that

$$|I_{21} - I_{21}'| \leqslant C\varepsilon^4(\boldsymbol{EM}\,\tilde{\Psi}^{3/2})^{2/3}, \tag{61}$$

for some constant $C > 0$. Here I_{21}' is defined by a formula analogous to Eq. (57) with the replacement of $\tilde{W}_{p_1\cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N)$ in the kth term by $\tilde{W}_{p_1\cdots p_r}^{1,\varrho_{k-1}}(\varrho_N, t_2, \ldots, t_r, w_N)$.

This follows from the fact that $\varrho_k \leqslant \varrho_N - \varepsilon^{\gamma}$ for $k \leqslant N-1$. By virtue of Lemma 2, we obtain that there exists $C>0$ for which $\|U^0_{\varepsilon,p}(\varrho_N,\varrho_k,0)\|_{L^2} \leqslant C\varepsilon^4$, $k \leqslant N-1$.

Let us consider now the following expression:

$$I_3 = \sum_{k=1}^{N} \sum_{m_k \cdots m_N = 0} \int_{(\mathbb{R}^d)^N} \mathscr{K}_k \mathscr{L}_k(w_{k-1},\ldots,w_N)\,\mathrm{d}w^{(N)}. \tag{62}$$

Here

$$\mathscr{K}_k = \boldsymbol{ME}\left[\prod_{l=1}^{k-1} \mathscr{Q}_{m_l}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1})\,\tilde{\Psi}\right]$$

and

$$\begin{aligned}\mathscr{L}_k(w_{k-1},\ldots,w_N) = \boldsymbol{E}\Bigg\{ & p_\varepsilon^{0,\varrho_{k-1}}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}) \\ & \times \prod_{l=k+1}^{N} [p_\varepsilon^{1,\varrho_{l-1}}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1}) - p_\varepsilon^{0,\varrho_{l-1}}(\varrho_l, w_l, \varrho_{l-1}, w_{l-1})] \\ & \times \tilde{W}_{p_1\cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N)\Bigg\}.\end{aligned}$$

Spatial homogeneity of $(\boldsymbol{U}^0_\varepsilon, \boldsymbol{U}^1_\varepsilon, \boldsymbol{U}_\varepsilon)$ implies that

$$\mathscr{L}_k(w_{k-1},\ldots,w_N) = \mathscr{L}_k(w_{k-1}-w_N,\ldots,w_{N-1}-w_N,0).$$

Let us introduce a change of variables $w'_k = w_k - w_N, \ldots, w'_{N-1} = w_{N-1} - w_N$, $w'_N = w_N$. Since $\boldsymbol{V}$ is of divergence free, we have $\int p_\varepsilon^{\theta,u}(t,x;s,z)\,\mathrm{d}z = \int p_\varepsilon^{\theta,u,}(t,z;s,y)\,\mathrm{d}z = 1$ for all $0 \leqslant \theta \leqslant 1, x, y \in \mathbb{R}^d$. We get, upon integration with respect to w'_N, that all the terms of the sum (62) corresponding to $k \leqslant N-1$ vanish. The term corresponding to $k=N$ also vanishes thanks to the fact that $\boldsymbol{E}\,\tilde{W}_{p_1\cdots p_r}(\varrho_N, t_2, \ldots, t_r, 0) = 0$. Hence $I_3 = 0$.

On the other hand, expanding each term $p_\varepsilon^{\theta,\varrho_{k-1}}$ using Eq. (19) we get that $I_3 = I_{30} + I_{31} + I_{32}$. Here

$$\begin{aligned}I_{30} = & \sum_{k=1}^{N} \sum_{m_1 \cdots m_N = 0}^{M} \int_{(\mathbb{R}^d)^N} \mathscr{K}_k \boldsymbol{E}\Bigg\{ \mathscr{Q}^{0,\varrho_{k-1}}_{m_k}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}) \\ & \times \prod_{l=k+1}^{N} \Delta\mathscr{Q}_{m_l}\,\tilde{W}_{p_1\cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N)\Bigg\}\,\mathrm{d}w^{(N)}, \\ I_{31} = & \sum_{k=1}^{N} \sum_{m_k,\ldots,m_N=0}^{M} \sum_{S,T} \sum_{m_{i_1},\ldots,m_{i_q}=0}^{M} \int_{(\mathbb{R}^d)^N} \mathscr{K}_k \boldsymbol{E}\Bigg\{ \mathscr{Q}^{0,\varrho_{k-1}}_{m_k}(\varrho_k, w_k, \varrho_{k-1}, w_{k-1}) \\ & \times \prod_{l=1}^{q} \Delta\mathscr{Q}_{m_{i_l}} \prod_{l=1}^{k-q-1} \Delta_{j_l}\mathscr{R}_M\,\tilde{W}_{p_1\cdots p_r}(\varrho_N, t_2, \ldots, t_r, w_N)\Bigg\}\,\mathrm{d}w^{(N)}.\end{aligned}$$

Here

$$\Delta_j \mathscr{R}_M = \mathscr{R}_M^{1,\varrho_{j-1}}(\varrho_j, w_j, \varrho_{j-1}, w_{j-1}) - \mathscr{R}_M^{0,\varrho_{j-1}}(\varrho_j, w_j, \varrho_{j-1}, w_{j-1}).$$

The summation $\sum_{S,T}$ extends over all partitions $S=\{i_1,\dots,i_q\}$, $T=\{j_1,\dots,j_{k-q-1}\}$ of the set $\{k+1,\dots,N\}$ of which set T is non-empty.

$$I_{32}=\sum_{k=1}^{N}\sum_{m_{k+1},\dots,m_N=0}^{M}\sum_{S,T}\sum_{m_{i_1},\dots,m_{i_q}=0}^{M}\int_{(\mathbb{R}^d)^N}\mathscr{K}_k\boldsymbol{E}\left\{\mathscr{R}_M^{0,\varrho_{k-1}}(\varrho_k,w_k,\varrho_{k-1},w_{k-1})\right.$$
$$\left.\times\prod_{l=1}^{q}\Delta\mathscr{Q}_{m_{i_l}}\prod_{l=1}^{k-q-1}\Delta_{j_l}\mathscr{R}_M\,\tilde{W}_{p_1\cdots p_r}(\varrho_N,t_2,\dots,t_r,w_N)\right\}\mathrm{d}w^{(N)}.$$

Here the summation $\Sigma_{S,T}$ is as in the definition of I_{31} although at this time we admit also T to be an empty set. Using Remark 4 we get that

$$|I_{31}|+|I_{32}|\leqslant C\varepsilon^{2\gamma'}(\boldsymbol{E}\boldsymbol{M}\tilde{\Psi}^{3/2})^{2/3}$$

for some $C>0$ and $2>\gamma'>\gamma$, under our choice of M.

Claim. *We have*

$$|I'_{21}-I_{30}|\leqslant C\varepsilon^4(\boldsymbol{E}\boldsymbol{M}\,\tilde{\Psi}^{3/2})^{2/3}. \tag{63}$$

Let us observe that $\mathscr{Q}_{m_j}^{1,\varrho_{j-1}}=\mathscr{Q}_{m_j}^{1,\varrho_{k-1}}$ for $j\geqslant k+1$. We can write that

$$\mathscr{Q}_{m_j}^{1,\varrho_{j-1}}=\mathscr{Q}_{m_j}^{0,\varrho_{k-1}}+\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\mathscr{Q}_{m_j}^{\theta,\varrho_{k-1}}\,\mathrm{d}\theta. \tag{64}$$

Hence, substituting for $\mathscr{Q}_{m_j}^{1,\varrho_{j-1}}$, $j=k+1,\dots,N$ into the expression defining I'_{21} and subsequently using Lemma 1 we obtain $I'_{21}=\mathscr{C}+\mathscr{R}$, where

$$\mathscr{C}=\sum_{k=1}^{N}\sum_{m_1\cdots m_N=0}^{M}\int_{(\mathbb{R}^d)^N}\mathscr{K}_k\boldsymbol{E}\left\{\mathscr{Q}_{m_k}^{0,\varrho_{k-1}}(\varrho_k,w_k,\varrho_{k-1},w_{k-1})\prod_{l=k+1}^{N}\Delta'\mathscr{Q}_{m_l}\right.$$
$$\left.\times\tilde{W}_{p_1\cdots p_r}^{1,\varrho_k}(\varrho_N,t_2,\dots,t_r,w_N)\right\}\mathrm{d}w^{(N)}$$

and

$$\Delta'\mathscr{Q}_{m_j}=\mathscr{Q}_{m_j}^{0,\varrho_{k-1}}(\varrho_j,w_j,\varrho_{j-1},w_{j-1})-\mathscr{Q}_{m_j}^{0,\varrho_{j-1}}(\varrho_j,w_j,\varrho_{j-1},w_{j-1}).$$

On the other hand,

$$\mathscr{R}=\sum_{k=1}^{N}\sum_{m_1\cdots m_N=0}^{M}\sum_{S,T}\int_{[0,1]^q}\int_{(\mathbb{R}^d)^N}\boldsymbol{E}\boldsymbol{M}\left\{\prod_{l=1}^{k-1}\mathscr{Q}_{m_l}(\varrho_l,w_l,\varrho_{l-1},w_{l-1})\right.$$
$$\times\mathscr{Q}_{m_k}^{0,\varrho_{k-1}}(\varrho_k,w_k,\varrho_{k-1},w_{k-1})\prod_{l=1}^{q}\frac{\mathrm{d}}{\mathrm{d}\theta_{i_l}}\mathscr{Q}_{m_{i_l}}^{\theta_{i_l},\varrho_{k-1}}(\varrho_{i_l},w_{i_l},\varrho_{i_l-1},w_{i_l-1})$$
$$\left.\times\prod_{l=1}^{k-q-1}\Delta'\mathscr{Q}_{m_{j_l}}\,\tilde{W}_{p_1\cdots p_r}^{1,\varrho_k}(\varrho_N,t_2,\dots,t_r,w_N)\tilde{\Psi}\right\}\mathrm{d}\theta_{i_1\cdots\mathrm{d}\theta_{i_q}}\,\mathrm{d}w^{(N)} \tag{65}$$

and $\Sigma_{S,T}$ extends over all disjoint subsets $S=\{i_1,\dots,i_q\}$, $T=\{j_1,\dots,j_{k-q-1}\}$ of the set $\{k+1,\dots,N\}$ such that S is non-empty.

After performing the differentiations $\mathrm{d}/\mathrm{d}\theta_i$ we can see that the terms which constitute the sum we called $\mathscr{R}$ are of the form of products whose at least one factor is of the form either $\boldsymbol{U}_\varepsilon^0(\varrho,\varrho_{k-1},x)-\boldsymbol{U}_\varepsilon^0(\tau,\varrho_{k-1},y)$, or $1/\varepsilon\int_{\tau_1}^{\tau_2}\boldsymbol{U}_\varepsilon^0(\varrho,\varrho_{k-1},x)\,\mathrm{d}\varrho$ for $\varrho_{i_l-1}\leqslant\tau_1\leqslant\tau_2\leqslant\varrho_{i_l}$, $x\in\mathbb{R}^d$. Remembering that $\varrho_{i_l}-\varrho_{k-1}\geqslant\varepsilon^\gamma$ for all i_l-s we can see, upon an application of Lemma 2, that L^p norms of these factors are smaller than $C\varepsilon^4$. Using Remarks 4 and 5 we can estimate $\mathscr{R}$ in Eq. (65) by $C\varepsilon^4(\boldsymbol{EM}\,\tilde{\Psi}^{3/2})^{2/3}$.

As for $\mathscr{C}$ we can use again the expression (64) to substitute for $\mathscr{Q}_{m_j}^{0,\varrho_{k-1}}(\varrho_j,w_j,\varrho_{j-1},w_{j-1})$ into Eq. (65) getting $\mathscr{C}=I_{30}+\mathscr{R}'$, where $\mathscr{R}'$ is an expression of the form analogous with $\mathscr{R}$. $\mathscr{R}'$ can be estimated in the same fashion by $C\varepsilon^4(\boldsymbol{EM}\,\tilde{\Psi}^{3/2})^{2/3}$. □

Proof of Lemma 5. Notice that

$$|\boldsymbol{EM}[\Delta X_p(t')\Delta X_p(t)\Psi]|=\frac{1}{\varepsilon}\int_{t'}^{t'+\varepsilon^\gamma}\boldsymbol{EM}[U_{\varepsilon,p}(\varrho,X(\varrho))\Psi_0]\,\mathrm{d}\varrho,$$

where $\Psi_0=\Delta X_p(t)\Psi$ is $\tilde{\mathscr{V}}_t$-measurable. Thanks to Lemma 12 we get upon an application of Hölder inequality that

$$|\boldsymbol{EM}\{\Delta X_p(t')\Psi_0|\leqslant C\varepsilon^{2\gamma'}(\boldsymbol{EM}|\Psi_0|^{3/2})^{2/3}\leqslant C\varepsilon^{5/2\gamma'}(\boldsymbol{EM}\,\Psi^6)^{1/6}\tag{66}$$

for γ' as in the statement of the lemma. This ends the proof of Lemma 5. □

Proof of Lemma 8. Consists in a direct application of Lemma 12. □

Proof of Lemma 9. At first we choose γ_0 as in Lemma 7. We shall adjust it further during the course of the proof. Let $\gamma\geqslant\gamma_0$. Estimating precisely as in Lemma 7 we can write that

$$|\boldsymbol{EM}\{\Delta X_p(t')\Delta X_q(t')\Psi\}-\varepsilon^\gamma(2\delta_{pq}+G_{pq}+G_{qp})\boldsymbol{EM}\,\Psi|\leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\,\Psi^4)^{1/4}\tag{67}$$

for a certain $\gamma'>\gamma$. Here

$$G_{pq}=\frac{1}{\varepsilon^2}\int_0^{\varepsilon^\gamma}\mathrm{d}\tau\int_0^\tau\boldsymbol{EM}\{\boldsymbol{M}[U_{\varepsilon,p}(\varrho,X(\varrho))]V_q(0,0)\Psi_{\varrho,\tau}\}\,\mathrm{d}\varrho,$$

where $\Psi_{\varrho,\tau}(\omega,\xi)$ is a certain non-negative, $\mathscr{V}_{-\infty,0}\otimes\mathscr{Z}_{(t'+\tau-\varrho)/\varepsilon^2}$-measurable random variable such that $\boldsymbol{EM}\,\Psi_{\varrho,\tau}^p\leqslant\boldsymbol{EM}\,\Psi^p$, for all $p\geqslant1,\tau\geqslant\varrho\geqslant0$. Recalling the estimates we have made during the course of the proof of Lemma 7 we can write that there exist constants $C>0$ and $2>\gamma'>\gamma$ such that $|G_{pq}-G'_{pq}|\leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\,\Psi^2)^{1/2}$, where

$$G'_{pq}=\frac{1}{\varepsilon^2}\int_0^{\varepsilon^\gamma}\mathrm{d}\tau\int_0^\tau\mathrm{d}\varrho\int_{\mathbb{R}^d}\boldsymbol{EM}[q_\varepsilon(\varrho,x,0,0)U_{\varepsilon,p}(\varrho,x)V_q(0,0)\Psi_{\varrho,\tau}]\,\mathrm{d}x.\tag{68}$$

This can be reduced further by writing

$$|G'_{pq}-G''_{pq}|\leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^2)^{1/2}, \tag{69}$$

where G''_{pq} is defined by Eq. (68), in which $U_{\varepsilon,p}(\varrho,x)$ is replaced by $U_{\varepsilon,p}(\varrho,0)$. Indeed, using Hölder inequality we can write that

$$|G'_{pq}-G''_{pq}|\leqslant \frac{C}{\varepsilon^2}(\boldsymbol{EM}\Psi^6)^{1/6}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\varrho\int_{\mathbb{R}^d}$$
$$\times\{\boldsymbol{E}[q_\varepsilon(\varrho,x,0,0)]^6\}^{1/6}\{\boldsymbol{E}[U_{\varepsilon,p}(\varrho,x)-U_{\varepsilon,p}(\varrho,0)]^2\}^{1/2}\,\mathrm{d}x. \tag{70}$$

Using Lemma 3 and estimate (41) we shall obtain that

$$|G'_{pq}-G''_{pq}|\leqslant \frac{C}{\varepsilon^{2+\alpha}}(\boldsymbol{EM}\Psi^6)^{1/6}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\sqrt{\varrho}\,\mathrm{d}\varrho\int_{\mathbb{R}^d}\frac{|x|}{\sqrt{\varrho}}q(\Lambda\varrho,x)\,\mathrm{d}x$$
$$\leqslant \frac{C}{\varepsilon^{2+\alpha}}(\boldsymbol{EM}\Psi^6)^{1/6}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\varrho\int_{\mathbb{R}^d}q(2\Lambda\varrho,x)\sqrt{\varrho}\,\mathrm{d}x$$

for a certain sufficiently large Λ. Integrating over x we get that $|G'_{pq}-G''_{pq}|\leqslant C\varepsilon^{5/2\gamma-2-\alpha}$ $(\boldsymbol{EM}\Psi^6)^{1/6}$, which guarantees (69). Finally, we claim that

$$|G''_{pq}-G^0_{pq}|\leqslant C\varepsilon^{\gamma'}(\boldsymbol{EM}\Psi^2)^{1/2}, \tag{71}$$

where

$$G^0_{pq}=\frac{2}{\varepsilon^2}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\boldsymbol{EM}[U_{\varepsilon,p}(\varrho,0)V_q(0,0)\Psi_{\varrho,\tau}]\,\mathrm{d}\varrho.$$

Let us denote by $q_\varepsilon^{(\theta)}(t,x,s,y;r)$ the expression defined by Eq. (16) with the drift $\boldsymbol{b}(t,x)$ replaced by $\frac{1}{\varepsilon}[(1-\theta)\boldsymbol{U}_\varepsilon(t,0)+\theta\boldsymbol{U}_\varepsilon(t,x)]$ and $\boldsymbol{Q}_\varepsilon^{(\theta)}=\nabla_x q_\varepsilon^{(\theta)}$. Notice that $q_\varepsilon^{(1)}=q_\varepsilon$. We can write

$$G''_{pq}=G^0_{pq}+\frac{1}{\varepsilon^2}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\varrho\int_0^1\mathrm{d}\theta\int_{\mathbb{R}^d}\boldsymbol{EM}$$
$$\times\left[\frac{\mathrm{d}}{\mathrm{d}\theta}q_\varepsilon^{(\theta)}(\varrho,x,0,0)U_{\varepsilon,p}(\varrho,0)V_p(0,0)\Psi_{\varrho,\tau}\right]\mathrm{d}x$$
$$=G^0_{pq}+\frac{1}{\varepsilon^2}\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\varrho\int_0^1\mathrm{d}\theta\int_{\mathbb{R}^d}\mathrm{d}x$$
$$\times\boldsymbol{EM}\left[\frac{1}{\varepsilon}\int_0^{\varrho}(\boldsymbol{U}_\varepsilon(\tau',x)-\boldsymbol{U}_\varepsilon(\tau',0))\cdot\boldsymbol{Q}_\varepsilon^{(\theta)}(\varrho,x,0,0;1)\,\mathrm{d}\tau'\,U_{\varepsilon,p}(\varrho,0)V_q(0,0)\Psi_{\varrho,\tau}\right].$$

Estimating as in (70) with the help of Lemmas 11, 3 and (41) we shall obtain that $|G''_{pq}-G^0_{pq}|\leqslant C\varepsilon^{3\gamma-\alpha-3}$, which guarantees the desired bound if only γ_0 is chosen bigger than $(\alpha+3)/2$.

Using condition (2) of Lemma 10 we can write that

$$G^0_{pq}=\frac{1}{\varepsilon^2}\int_{t'}^{t'+\varepsilon^{\gamma}}\mathrm{d}\tau\int_{t'}^{\tau}\boldsymbol{EM}[U_{\varepsilon,p}(\tau,X(\tau'))U_{\varepsilon,q}(\tau',X(\tau'))\Psi]\,\mathrm{d}\tau'$$

$$=\frac{1}{\varepsilon^2}\int_{t'}^{t'+\varepsilon^{\gamma}}\mathrm{d}\tau\int_{t'}^{\tau}\boldsymbol{EM}[\tilde{W}_{p,q}(\tau,\tau',X(\tau'))\Psi]\,\mathrm{d}\tau'$$

$$+\frac{1}{\varepsilon^2}\int_{t'}^{t'+\varepsilon^{\gamma}}\mathrm{d}\tau\int_{t'}^{\tau}\bar{W}_{p,q}(\tau,\tau')\boldsymbol{EM}\Psi\,\mathrm{d}\tau'.$$

Here $\bar{W}_{p,q}(t,s)=\boldsymbol{E}W_{p,q}(t,s,0)$. Applying Lemma 12 we get that

$$|\boldsymbol{EM}[\tilde{W}_{p,q}(\tau,\tau',X(\tau'))\Psi]|\leqslant C\varepsilon^{2\gamma'}(\boldsymbol{EM}\Psi^{3/2})^{2/3}.$$

The proof of our lemma is then a conclusion from the following estimate:

$$\left|\left[\frac{1}{\varepsilon^2}\int_{t'}^{t'+\varepsilon^{\gamma}}\mathrm{d}\tau\int_{t'}^{\tau}\bar{W}_{p,q}(\tau,\tau')\,\mathrm{d}\tau'-\varepsilon^{\gamma}\int_0^{+\infty}\boldsymbol{E}[V_p(\varrho,0)V_q(0,0)]\,\mathrm{d}\varrho\right]\boldsymbol{EM}\Psi\right|$$

$$\leqslant\int_0^{\varepsilon^{\gamma}}\mathrm{d}\tau\int_{\frac{\tau}{\varepsilon^2}}^{+\infty}\frac{\mathrm{d}\tau'}{1+\tau'4}\boldsymbol{EM}\Psi\leqslant\varepsilon^2\boldsymbol{EM}\Psi.\qquad\square$$

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